Problem 1: Suppose T:R2R4, T(e1) = (1, 2, 3, 4) And T(e2) =(5, 5, 0, 0), Where E1=(1, 0)and E2=(0,1).

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Introduction to Linear Transformations

Linear transformations are fundamental concepts in linear algebra, serving as functions that map vectors from one vector space to another while preserving vector addition and scalar multiplication. In this problem, we analyze a specific linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^4$ with known images of standard basis vectors. Understanding such transformations is crucial in various applications, including computer graphics, data science, and engineering.

This article explores the detailed process of determining the transformation T, its matrix representation, properties, and implications based on the given data.

Understanding the Given Data

The problem states:

- $T: \mathbb{R}^2 \to \mathbb{R}^4$
- $T(e_1) = (1, 2, 3, 4)$
- $T(e_2) = (5, 5, 0, 0)$

where

- $e_1 = (1, 0)$
- $e_2 = (0, 1)$

Key points to note:

- e_1 and e_2 are the standard basis vectors in $\mathbb{R}^2$.
- The images of these basis vectors under T are provided, which is typical for defining linear transformations.
- The goal is to find the matrix representation of T, analyze its properties, and understand how it acts on arbitrary vectors in $\mathbb{R}^2$.

Matrix Representation of the Transformation T

In linear algebra, the transformation T can be represented by a matrix A such that:

$$
T(\mathbf{x}) = A \mathbf{x}
$$

for any $\mathbf{x} \in \mathbb{R}^2$.

Since the images of the basis vectors are known, the columns of A are precisely these images:

$$
A = \left[T(e_1) \quad T(e_2) \right]
$$

which becomes:

$$
A = \begin{bmatrix}
1 & 5 \\
2 & 5 \\
3 & 0 \\
4 & 0
\end{bmatrix}
$$

This matrix explicitly captures the transformation T, allowing us to compute the image of any vector in $\mathbb{R}^2$.

Calculating the Transformation of Arbitrary Vectors

Any vector $\mathbf{x} \in \mathbb{R}^2$ can be written as:

$$
\mathbf{x} = x_1 e_1 + x_2 e_2 = (x_1, x_2)
$$

Applying T:

$$
T(\mathbf{x}) = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 T(e_1) + x_2 T(e_2)
$$

which simplifies to:

$$
T(\mathbf{x}) = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \end{bmatrix}
$$

or component-wise:

$$
T(\mathbf{x}) = \begin{bmatrix}
x_1 \cdot 1 + x_2 \cdot 5 \\
x_1 \cdot 2 + x_2 \cdot 5 \\
x_1 \cdot 3 + x_2 \cdot 0 \\
x_1 \cdot 4 + x_2 \cdot 0
\end{bmatrix}
$$

which simplifies to:

$$
T(\mathbf{x}) = \begin{bmatrix}
x_1 + 5x_2 \\
2x_1 + 5x_2 \\
3x_1 \\
4x_1
\end{bmatrix}
$$

This provides a concrete way to find the image of any vector in $\mathbb{R}^2$.

Properties of the Transformation T

Understanding the properties of T is essential for applications and theoretical insights. Key properties include linearity, rank, nullity, and invertibility.

Linearity

Given that T is defined via a matrix multiplication, it inherently satisfies linearity:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- $T(c \mathbf{x}) = c T(\mathbf{x})$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and scalar c.

Rank and Nullity

The rank of T corresponds to the rank of matrix A:

$$
\operatorname{rank}(A) = \text{number of linearly independent columns}
$$

Checking if the columns are linearly independent:

- The first column: $(1, 2, 3, 4)$
- The second column: $(5, 5, 0, 0)$

Are these vectors linearly independent?

Suppose c_1 and c_2 satisfy:

$$
c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}
$$

which leads to the system:

$$
\begin{cases}
c_1 + 5 c_2 = 0 \\
2 c_1 + 5 c_2 = 0 \\
3 c_1 + 0 c_2 = 0 \\
4 c_1 + 0 c_2 = 0
\end{cases}
$$

From the last two equations:

- $3 c_1 = 0 \Rightarrow c_1 = 0$
- $4 c_1 = 0 \Rightarrow c_1 = 0$

Substitute into the first equation:

$$
0 + 5 c_2 = 0 \Rightarrow c_2 = 0
$$

Thus, only the trivial solution exists, implying the columns are linearly independent, and:

$$
\operatorname{rank}(A) = 2
$$

Since the rank equals the dimension of the domain ($\mathbb{R}^2$), T is injective (one-to-one). However, the rank is less than the codomain dimension (which is 4), so T is not surjective (onto).

- Nullity (dimension of the kernel):

$$
\text{Nullity} = \dim(\mathbb{R}^2) - \operatorname{rank}(A) = 2 - 2 = 0
$$

meaning the kernel contains only the zero vector.

Invertibility

Since T is from $\mathbb{R}^2$ to $\mathbb{R}^4$, and its matrix A has full column rank, T is not invertible as a function from $\mathbb{R}^2$ to $\mathbb{R}^4$, because invertibility requires a square matrix. However, the transformation is injective, which means it preserves distinctness of vectors in the domain.

Geometric Interpretation of T

While T maps from a 2D space to a 4D space, we can interpret its effect on vectors:

- The images of basis vectors define a plane in $\mathbb{R}^4$.
- The transformation stretches, shears, or translates the input vectors depending on the matrix A.

Specifically:

- $T(e_1) = (1, 2, 3, 4)$: the image of e_1 in 4D space.
- $T(e_2) = (5, 5, 0, 0)$: the image of e_2.

Any vector in $\mathbb{R}^2$ is mapped onto a linear combination of these two vectors, forming a 2D subspace within $\mathbb{R}^4$.

Applications and Significance of the Transformation \(T \)

Understanding this transformation has practical implications:

- Data Dimensionality Reduction: Mapping 2D data into 4D for analysis.
-

Frequently Asked Questions

What is the linear transformation T from R^2 to R^4 defined as in Problem 1?

T is a linear transformation from R^2 to R^4, with $T(e_1) = (1, 2, 3, 4)$ and $T(e_2) = (5, 5, 0, 0)$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

How can we find the matrix representation of T based on the given information?

The matrix representation of T is formed by placing $T(e_1)$ and $T(e_2)$ as columns, resulting in a 4×2 matrix:
[[1, 5],
[2, 5],
[3, 0],
[4, 0]].

What is the matrix form of T in standard basis?

The matrix form of T in standard basis is:
[[1, 5],
[2, 5],
[3, 0],
[4, 0]].

How do you compute T for an arbitrary vector (x, y) in R^2?

To compute T(x, y), multiply the matrix by the vector:
$T(x, y) = x\,T(e_1) + y\,T(e_2) = (x + 5y, 2x + 5y, 3x, 4x)$.

Is T a linear transformation, and how can we verify this?

Yes, T is linear. It can be verified because it satisfies linearity properties: $T(u + v) = T(u) + T(v)$ and $T(cu) = cT(u)$ for all vectors u, v and scalars c.

What is the image (range) of T?

The image of T is the span of the vectors $T(e_1)$ and $T(e_2)$, which are (1, 2, 3, 4) and (5, 5, 0, 0).

How can we determine if T is injective (one-to-one)?

T is injective if the columns of its matrix are linearly independent. Since the columns are not scalar multiples and are linearly independent, T is injective.

What is the kernel of T?

The kernel consists of all vectors (x, y) in R^2 such that T(x, y) = (0, 0, 0, 0). Solving for x and y, we find the kernel is { (x, y) | x + 5y = 0, 2x + 5y = 0, 3x = 0, 4x = 0 } which implies x=0 and y=0, so the kernel is trivial.

How would you find the rank of the transformation T?

The rank of T is the dimension of its image, which is the number of linearly independent columns. Since the columns are independent, rank(T) = 2.

Additional Resources

Analyzing the Transformation \(T: \mathbb{R}^2 \to \mathbb{R}^4\) with Given Basis Images: A Deep Dive into Linear Transformations

Understanding linear transformations forms a cornerstone of linear algebra, as they bridge the gap between algebraic operations and geometric intuition. This article offers a comprehensive exploration of a specific transformation \(T\), which maps vectors from a two-dimensional real vector space \(\mathbb{R}^2\) into a four-dimensional space \(\mathbb{R}^4\). With given images of basis vectors, the discussion will dissect the nature of \(T\), its matrix representation, and the implications for the structure of the transformation. Such an analysis not only sheds light on the mechanics of this particular transformation but also exemplifies broader principles applicable across linear algebra.

Understanding the Setup: The Transformation \(T\) and Its Given Data

The Domain and Codomain

The function \(T\) is defined as a linear transformation from \(\mathbb{R}^2\) to \(\mathbb{R}^4\):

\[
T: \mathbb{R}^2 \to \mathbb{R}^4
\]

This indicates that each vector in the two-dimensional plane is mapped to a four-dimensional vector. The nature of this transformation, whether it is stretching, compressing, rotating, or skewing, depends on the images of basis vectors and the transformation's matrix representation.

Basis Vectors and Their Images

The standard basis for \(\mathbb{R}^2\) is:

\[
E_1 = (1, 0), \quad E_2 = (0, 1)
\]

The transformation \(T\) acts on these basis vectors as follows:

\[
T(E_1) = (1, 2, 3, 4)
\]
\[
T(E_2) = (5, 5, 0, 0)
\]

These images are fundamental because, in linear algebra, the entire transformation can be reconstructed from the images of basis vectors. This property facilitates the matrix representation of \(T\).

Constructing the Matrix Representation of \(T\)

Matrix Columns Corresponding to Basis Vectors

In the context of linear transformations, the matrix representation \(A\) of \(T\) with respect to the standard bases is constructed by placing the images of the basis vectors as columns:

\[
A = \begin{bmatrix}
| & | \\

T(E_1) & T(E_2) \\
| & |
\end{bmatrix}
= \begin{bmatrix}
1 & 5 \\
2 & 5 \\
3 & 0 \\
4 & 0
\end{bmatrix}
\]

This matrix succinctly encapsulates the action of \(T\):

\[
T(\mathbf{x}) = A \mathbf{x}
\]

for any \(\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2\).

Implications of the Matrix Form

The matrix \(A\) tells us exactly how the transformation acts:

- The first column indicates the image of \(E_1\), revealing how the transformation stretches or skews along that vector.
- The second column shows the effect on \(E_2\).

Analyzing this matrix can provide insights into properties like rank, nullity, invertibility, and the geometric nature of the transformation.

Analyzing the Transformation: Properties and Geometric Intuition

Rank and Dimension of the Image

The rank of the matrix \(A\) determines the dimension of the image (range) of \(T\):

\[
A = \begin{bmatrix}
1 & 5 \\

2 & 5 \\
3 & 0 \\
4 & 0
\end{bmatrix}
\]

Calculating the rank involves examining linear independence among columns:

- The columns are \((1, 2, 3, 4)^T\) and \((5, 5, 0, 0)^T\).

Check if they are linearly independent:

Suppose \(c_1 T(E_1) + c_2 T(E_2) = 0\), i.e.,

\[
c_1 (1, 2, 3, 4) + c_2 (5, 5, 0, 0) = (0, 0, 0, 0)
\]

which translates to the system:

\[
\begin{cases}
c_1 + 5 c_2 = 0 \\
2 c_1 + 5 c_2 = 0 \\
3 c_1 + 0 c_2 = 0 \\
4 c_1 + 0 c_2 = 0
\end{cases}
\]

From the third and fourth equations:

\[
3 c_1 = 0 \Rightarrow c_1 = 0
\]
\[
4 c_1 = 0 \Rightarrow c_1 = 0
\]

Substituting \(c_1=0\) into the first:

\[
0 + 5 c_2 = 0 \Rightarrow c_2 = 0
\]

Thus, the only solution is \(c_1 = c_2 = 0\), indicating the columns are linearly independent.

Conclusion: The rank of \(A\) is 2, meaning the image of \(T\) is a 2-dimensional subspace of \(\mathbb{R}^4\). This is expected, as the domain is 2-dimensional, and the transformation is likely to be injective onto its image.

Kernel (Null Space) of T

Since the columns are linearly independent, the kernel contains only the zero vector:

$$
\ker T = \{\mathbf{0}\}
$$

Hence, T is injective (one-to-one). This property is significant because it indicates that T preserves the distinctness of vectors in $\mathbb{R}^2$, just embedded into a higher-dimensional space.

Range (Image) of T

The image of T is a 2-dimensional subspace in $\mathbb{R}^4$. To understand its nature geometrically, consider the span of the two image vectors:

$$
\operatorname{Im}(T) = \operatorname{span}\left\{ (1, 2, 3, 4), (5, 5, 0, 0) \right\}
$$

These vectors form a basis for the image, and any vector in the image can be expressed as a linear combination:

$$
\mathbf{v} = \alpha (1, 2, 3, 4) + \beta (5, 5, 0, 0)
$$

This subspace is embedded within $\mathbb{R}^4$ and can be visualized as a plane passing through the origin, spanned by these basis vectors.

Deeper Insights: Geometric and Algebraic Significance

Transformation's Geometric Effect

While $\mathbb{R}^2$ is a plane, the transformation maps this plane into a 2D subspace within $\mathbb{R}^4$. The images of the standard basis vectors reveal how the transformation "embeds" the plane into higher dimensions:

- E_1 is mapped to $(1, 2, 3, 4)$, indicating a certain stretching and skewing.
- E_2 is mapped to $(5, 5, 0, 0)$, which suggests a different orientation and scaling.

The combination of these effects showcases the linear transformation's ability to distort the original plane in various ways—stretching along certain directions, compressing along others, and skewing the shape within the higher-dimensional space.

Implications for Invertibility and Isomorphism

Since the transformation's matrix A has rank 2 (full rank relative to the domain), T is injective, but not surjective onto $\mathbb{R}^4$. The image is only a 2D subspace, not the entire 4D space. Therefore:

- Not invertible as a map from $\mathbb{R}^2$ to $\mathbb{R}^4$.
- Injective but not surjective.

This aligns with the general principle that a linear transformation from a lower-dimensional space to a higher-dimensional space cannot be surjective unless the codomain is restricted accordingly.

Applications and Broader Context

Understanding Through Examples

Transformations like T are common in various fields:

- Data Science and Machine Learning: Embedding lower-dimensional data into higher-dimensional feature spaces to facilitate separation or classification.
- Computer Graphics: Mapping 2D images into 4D space for transformations in higher

[Problem 1 Suppose T R2r4 T E1 1 2 3 4 And T E2 5 5 0 0 Where E1 1 0 And E2 0 1]

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