Problem 5 (50 Points) Determine Whether The Given Linear Transformation Is Invertible. 1(x, X, X3, X)

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Understanding whether a linear transformation is invertible is a fundamental concept in linear algebra, with significant implications in various fields such as engineering, computer science, physics, and mathematics. In this article, we will explore the process of determining the invertibility of a specific linear transformation, denoted as T, defined by the mapping T: P → P, where P is a polynomial space, and T is given by T(f(x)) = f'(x) + f(x^3) + f(x). The goal is to analyze this transformation thoroughly and establish whether it possesses an inverse.

What Is a Linear Transformation?

Before delving into the specifics of the problem, it is crucial to understand what a linear transformation entails.

Definition of a Linear Transformation

A linear transformation T between two vector spaces V and W over the same field F is a function T: V → W satisfying two properties for all vectors u, v ∈ V and scalars c ∈ F:

1. Additivity: T(u + v) = T(u) + T(v)
2. Scalar Multiplication: T(cu) = cT(u)

These properties ensure that the transformation preserves the structure of the vector space, such as vector addition and scalar multiplication.

Understanding the Given Transformation

The transformation in question is:

T(f(x)) = f'(x) + f(x^3) + f(x)

where f(x) is a polynomial function, and the transformation acts on the space of polynomials, typically denoted as P, which includes all polynomials with coefficients in a field like the real numbers.

Components of the Transformation

The transformation T combines three operations:

- Derivative: $f'(x)$, the derivative of $f(x)$
- Composition with Cubing: $f(x^3)$, the polynomial f evaluated at x^3
- Identity: $f(x)$, the original polynomial

This combination makes T a non-trivial linear operator, and determining its invertibility involves analyzing these components collectively.

Conditions for Invertibility of a Linear Transformation

A linear transformation $T: V \to V$ (from a vector space to itself) is invertible if and only if:

- It is bijective (both injective and surjective).
- Equivalently, the kernel of T contains only the zero vector, i.e., $\ker(T) = \{0\}$.
- The determinant of the associated matrix (if the transformation is represented in a basis) is non-zero.
- The transformation can be "reversed" via an inverse operator T^{-1} such that $T^{-1}(T(f)) = f$ for all f in P.

In infinite-dimensional spaces like polynomial spaces, the analysis often involves studying the kernel and the image of T rather than determinants, which are more applicable to finite-dimensional matrices.

Analyzing the Transformation T for Invertibility

Given the nature of T, the best approach is to analyze its kernel (null space). If the kernel contains only the zero polynomial, then T is injective, and considering the space is typically infinite-dimensional, injectivity often implies invertibility onto its image. If the image is the entire space, T is invertible.

Step 1: Find the Kernel of T

To determine whether T is invertible, we need to find all polynomials $f(x)$ such that $T(f) = 0$, i.e.,

$$f'(x) + f(x^3) + f(x) = 0$$

This functional equation must be satisfied by all x.

Step 2: Set Up the Functional Equation

The kernel condition:

$f'(x) + f(x^3) + f(x) = 0$

must hold for all x in the domain. Our goal is to find all polynomials f(x) satisfying this differential-functional equation.

Step 3: Explore Possible Solutions

Given the complexities, the analysis involves considering the degree of f(x):

- Suppose deg(f) = n
- The derivative f'(x) reduces the degree by one
- The term $f(x^3)$ increases the degree, especially for higher degrees
- The term f(x) maintains the degree

This suggests that the only polynomial solution might be the zero polynomial, but to confirm, we analyze specific cases.

Detailed Solution Strategy

Case 1: Polynomial of degree 0 (Constant Polynomial)

Let f(x) = c, where c is a constant.

Then:

$f'(x) = 0$

$f(x^3) = c$

$f(x) = c$

Equation:

$0 + c + c = 0 \rightarrow 2c = 0 \rightarrow c = 0$

Thus, the only constant polynomial in the kernel is the zero polynomial.

Case 2: Polynomial of degree 1 (Linear Polynomial)

Let $f(x) = ax + b$

Then:

$f'(x) = a$

$f(x^3) = a x^3 + b$

$f(x) = a x + b$

Equation:

$a + (a x^3 + b) + (a x + b) = 0$

Simplify:

$a + a x^3 + b + a x + b = 0$

Group like terms:

$a x^3 + a x + (a + 2b) = 0$

Since this polynomial must be zero for all x, each coefficient must be zero:

- Coefficient of x^3: $a = 0$
- Coefficient of x: $a = 0$ (already zero)
- Constant term: $a + 2b = 0 \rightarrow 0 + 2b = 0 \rightarrow b = 0$

Hence, the only solution is $f(x) \equiv 0$.

Case 3: Polynomial of degree $n \geq 2$

Assuming $f(x) = \sum_{k=0}^n c_k x^k$

Then:

$f'(x) = \sum_{k=1}^n k c_k x^{k-1}$

$f(x^3) = \sum_{k=0}^n c_k x^{3k}$

$f(x) = \sum_{k=0}^n c_k x^k$

The functional equation:

$f'(x) + f(x^3) + f(x) = 0$

becomes:

$$\sum_{k=1}^n k c_k x^{k-1} + \sum_{k=0}^n c_k x^{3k} + \sum_{k=0}^n c_k x^k = 0$$

Since the powers of x are different, for the sum to be zero for all x, the coefficients of each power must be zero separately.

This leads to a system of equations:

- For each power of x, the sum of coefficients must be zero.

Particularly, note that the degrees of the terms are:

- Derivative term: degree n-1
- $f(x^3)$: degrees 0, 3, 6, ..., 3n
- f(x): degrees 0, 1, 2, ..., n

The key observation is that these sets of degrees do not completely overlap unless the coefficients are zero, implying that the only polynomial solution is the zero polynomial.

Conclusion: The only polynomial in the kernel is the zero polynomial.

Implication: T Is Injective

Since the kernel contains only the zero polynomial, T is injective (one-to-one). In the context of polynomial spaces, injectivity often implies that T maps the space onto its image in a way that preserves the structure.

Is T Surjective? Does T Have an Inverse?

While injectivity is confirmed, invertibility requires T to be surjective (onto). For polynomial spaces, which are infinite-dimensional, the question reduces to whether T's image is the entire space.

Given the nature of T, which involves derivative and composition operations, it is generally not surjective over the entire polynomial space because:

- The derivative reduces degree.
- The composition with x^3 can increase degree significantly.
- The combination may not cover all polynomial functions.

Therefore, T is not necessarily surjective, and thus, it may not be invertible over the entire polynomial space.

However, if T is restricted to a suitable subspace where the image equals the entire subspace, T could be invertible on that subspace. But in the general setting, considering the entire polynomial space, T is not invertible.

Summary and Final Verdict

- The kernel of T contains only the zero polynomial, confirming injectivity.
- The transformation involves derivative and polynomial composition operations that do not guarantee surjectivity over the entire polynomial space.
- Therefore, T is not invertible over the entire space of polynomials.

Conclusion

The linear transformation T defined by $T(f(x)) = f'(x) + f(x^3) + f(x)$ is not invertible over the entire polynomial space because it is not surjective, despite being injective. This analysis underscores the importance of examining both the kernel and the image of a linear transformation to determine invertibility comprehensively.

Keywords: linear transformation, invertibility, polynomial

Frequently Asked Questions

What is the main goal when determining if the linear transformation $1(x, x, x^3, x)$ is invertible?

The main goal is to check whether the transformation has an inverse, which typically involves verifying if its associated matrix is invertible or if the transformation is one-to-one and onto.

How do you represent the linear transformation $1(x, x, x^3, x)$ in matrix form?

You evaluate the transformation on a basis and express the images as columns in a matrix; in this case, you analyze how the transformation acts on basis vectors to form the matrix.

What is the significance of the determinant of the matrix associated with the linear transformation in determining invertibility?

A non-zero determinant indicates the matrix (and thus the transformation) is invertible; a zero determinant means it is not invertible.

Does the presence of the cubic term x^3 affect the invertibility of the linear transformation?

Yes, because x^3 is a nonlinear term, which suggests the transformation may not be linear unless the context specifies a linear approximation or specific linear components.

Can the linear transformation $1(x, x, x^3, x)$ be considered linear given the presence of a cubic term?

No, because the inclusion of x^3 makes the transformation nonlinear, and linear transformations must satisfy additivity and homogeneity.

What conditions must be met for a linear transformation to be invertible in the context of polynomial mappings?

The transformation must be represented by a bijective linear operator, meaning its matrix must be invertible; polynomial mappings with nonlinear components typically are not linear unless specifically defined.

Is the function $1(x, x, x^3, x)$ a linear transformation from R to R^4?

No, because the x^3 term introduces nonlinearity, so the function is not a linear transformation.

How can one verify invertibility of a linear transformation involving polynomial terms?

By expressing the transformation as a matrix acting on a basis and checking if this matrix is invertible (i.e., has a non-zero determinant), but polynomial terms typically indicate nonlinear mappings.

Given the transformation $1(x, x, x^3, x)$, what is the conclusion regarding its invertibility?

Since it includes a cubic term, the transformation is nonlinear, and thus it cannot be invertible as a linear transformation unless specified as a different kind of mapping.

Additional Resources

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Introduction

In the realm of linear algebra, understanding the properties of linear transformations is fundamental, especially their invertibility. The problem at hand—determining whether the linear transformation $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$, defined by

$$
T(p(x)) = p(x) + x \cdot p'(x) + x^3 \cdot p''(x) + x,
$$

where $p(x) \in P_3(\mathbb{R})$ (the space of real polynomials of degree at most 3)—poses an intriguing question. Is T invertible? To answer this, we will systematically analyze the transformation's structure, its matrix representation, and properties such as injectivity and surjectivity.

This article aims to provide a comprehensive, step-by-step investigation into this problem, bridging theoretical insights with practical calculations, and ultimately delivering a conclusive answer. Such an exploration is not only academically enriching but also essential for students and researchers striving to master linear transformation concepts.

Theoretical Foundations of Linear Transformations and Invertibility

What Does It Mean for a Linear Transformation to Be Invertible?

A linear transformation $T: V \to W$ between vector spaces is invertible if there exists a linear transformation $T^{-1}: W \to V$ such that

$$
T^{-1}(T(v)) = v, \quad \text{for all } v \in V,
$$
and
$$
T(T^{-1}(w)) = w, \quad \text{for all } w \in W.
$$

In finite-dimensional spaces, invertibility is equivalent to:

- Injectivity (One-to-One): $\ker(T) = \{0\}$,
- Surjectivity (Onto): Range of T equals the entire codomain.

Equivalently, in finite-dimensional vector spaces, T is invertible if and only if its matrix representation (relative to some basis) is nonsingular, i.e., has a non-zero determinant.

Specification of the Transformation T

Let's formalize the given transformation:

$$
T(p(x)) = p(x) + x p'(x) + x^3 p''(x) + x,
$$

where $p(x) \in P_3(\mathbb{R})$.

Key observations:

- $p(x)$ is a polynomial of degree at most 3.
- $p'(x)$ is degree at most 2.
- $p''(x)$ is degree at most 1.
- The terms $p(x)$, $x p'(x)$, and $x^3 p''(x)$ are all polynomials of degree at most 3 (since multiplying a degree 2 polynomial by x yields degree 3; similarly for degree 1).

Thus, T maps $P_3(\mathbb{R})$ into itself.

Choosing a Basis and Constructing the Matrix Representation

To analyze invertibility, it's convenient to select a basis for $P_3(\mathbb{R})$. A standard basis is:

$$
\mathcal{B} = \{1, x, x^2, x^3\}.
$$

Any polynomial $p(x) \in P_3(\mathbb{R})$ can be written as

$$
p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.
$$

Our goal: determine the matrix $[T]_\mathcal{B}$ representing T with respect to this basis, then compute its determinant.

Step 1: Computing T on basis elements

For each basis element, compute T:

1. $p(x) = 1$:

- $p(x) = 1$,
- $p'(x) = 0$,
- $p''(x) = 0$.

Applying T:

\[

T(1) = 1 + x \cdot 0 + x^3 \cdot 0 + x = 1 + x.
\]

Expressed in basis form:

\[
T(1) = 1 + x = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3.
\]

2. \(p(x) = x \):

- \(p(x) = x \),
- \(p'(x) = 1 \),
- \(p''(x) = 0 \).

Applying \(T \):

\[
T(x) = x + x \cdot 1 + x^3 \cdot 0 + x = x + x + 0 + x = 3x.
\]

Expressed as:

\[
T(x) = 0 \cdot 1 + 3x + 0 \cdot x^2 + 0 \cdot x^3.
\]

3. \(p(x) = x^2 \):

- \(p(x) = x^2 \),
- \(p'(x) = 2x \),
- \(p''(x) = 2 \).

Applying \(T \):

\[
T(x^2) = x^2 + x \cdot 2x + x^3 \cdot 2 + x = x^2 + 2x^2 + 2x^3 + x.
\]

Combine like terms:

\[
T(x^2) = (x^2 + 2x^2) + x + 2x^3 = 3x^2 + x + 2x^3.
\]

In basis form:

\[
T(x^2) = 0 \cdot 1 + 1 \cdot x + 3 \cdot x^2 + 2 \cdot x^3.
\]

4. $p(x) = x^3$:

- $p(x) = x^3$,
- $p'(x) = 3x^2$,
- $p''(x) = 6x$.

Applying T:

$$
T(x^3) = x^3 + x \cdot 3x^2 + x^3 \cdot 6x + x = x^3 + 3x^3 + 6x^4 + x.
$$

Note that $6x^4$ is degree 4, outside P_3. Since $P_3(\mathbb{R})$ consists only of polynomials up to degree 3, we must interpret the transformation as acting within $P_3(\mathbb{R})$. Therefore, the presence of $6x^4$ indicates that the transformation, as defined, maps outside the space unless we clarify whether the transformation is restricted or projected.

Key Point: The initial problem states the transformation T is from $P_3(\mathbb{R})$ to itself. The inclusion of $x^3 p''(x)$ suggests that for polynomials of degree 3, derivatives can generate degree 2 and 1, but multiplying by x^3 can produce degree 4. To maintain the space, we interpret the action as restricting the image to degree at most 3, effectively truncating any higher degree terms.

Step 2: Handling the $x^3 p''(x)$ term

Given the above, for $p(x) = x^3$:

$$
p''(x) = 6x,
$$
and

$$
x^3 p''(x) = x^3 \cdot 6x = 6x^4.
$$

Since $6x^4 \notin P_3(\mathbb{R})$, the natural approach is to project the output onto $P_3(\mathbb{R})$. This aligns with the common practice in linear algebra to consider transformations between subspaces, where the image is restricted to the codomain.

Therefore, for the purpose of this problem:

$$
T(p(x)) := p(x) + x p'(x) + \text{proj}_{P_3}(x^3 p''(x)) + x,
$$

where proj_{P_3} indicates projection onto $P_3(\mathbb{R})$, i.e., discarding any degree higher than 3.

Applying this to \(p(x) = x^3 \):

\[
x^3 p''(x) = 6x^4,
\]
which is projected to zero in \(P_3 \). Thus,

\[
T(x^3) = x^3 + 3x^3 + 0 + x = 4x^3 + x.
\]

Expressed in basis:

\[
T(x^3) = 0 \cdot

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