Prove That A Function F Is Differentiable At X = A With F'(a)=b, BeR, If And Only If F(x)-f(a)-b(x-a)

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Understanding the differentiability of a function at a specific point is fundamental in calculus and analysis. This article provides a comprehensive explanation of the necessary and sufficient conditions for a function F to be differentiable at $x = a$ with derivative $F'(a) = b$, focusing on the equivalence involving the expression $F(x) - F(a) - b(x - a)$. We will explore the theoretical underpinnings, formal proof, implications, and applications of this important concept.

Introduction to Differentiability

What Does It Mean for a Function to Be Differentiable?

A function $F: \mathbb{R} \to \mathbb{R}$ is said to be differentiable at a point $a \in \mathbb{R}$ if the derivative $F'(a)$ exists at that point. The derivative at a intuitively measures the instantaneous rate of change of F at a, or equivalently, the slope of the tangent line to the graph of F at a.

Mathematically, F is differentiable at a if the following limit exists:

$$
F'(a) = \lim_{x \to a} \frac{F(x) - F(a)}{x - a}
$$

If this limit exists, we say that F is differentiable at a, and the value of the limit is $F'(a)$.

The Theorem: Characterization of Differentiability

Statement of the Theorem

The theorem provides a criterion for differentiability in terms of the behavior of the difference:

$$
F(x) - F(a) - b(x - a)
$$

\]

where b is a candidate for the derivative at a. The formal statement is:

> Theorem: A function F is differentiable at a with $F'(a) = b$ if and only if

$$
\lim_{x \to a} \frac{F(x) - F(a) - b(x - a)}{x - a} = 0
$$

which is equivalent to stating that

$$
F(x) - F(a) - b(x - a) = o(x - a) \quad \text{as} \quad x \to a
$$

meaning that the difference between $F(x)$ and its linear approximation at a vanishes faster than $x - a$ as x approaches a.

Intuitive Explanation

The expression $F(x) - F(a) - b(x - a)$ measures how well the linear function $L(x) = F(a) + b(x - a)$ approximates $F(x)$ near a. If this difference becomes negligible relative to $(x - a)$ as $x \to a$, then the function is well-approximated by its tangent line, indicating differentiability.

Conversely, if F is differentiable at a with derivative b, then the function behaves like its tangent line near a, and the difference $F(x) - F(a) - b(x - a)$ tends to zero faster than $(x - a)$.

Formal Proof of the Theorem

Necessity: If F is differentiable at a with $F'(a) = b$, then the limit holds

Suppose F is differentiable at a with derivative b. By the definition of differentiability:

$$
F'(a) = \lim_{x \to a} \frac{F(x) - F(a)}{x - a} = b
$$

This implies:

$$
\frac{F(x) - F(a)}{x - a} = b + \varepsilon(x)
$$

where $\varepsilon(x) \to 0$ as $x \to a$.

Multiplying both sides by $(x - a)$:

$$
F(x) - F(a) = b(x - a) + \varepsilon(x)(x - a)
$$

Rearranged, this becomes:

$$
F(x) - F(a) - b(x - a) = \varepsilon(x)(x - a)
$$

Dividing both sides by $(x - a)$:

$$
\frac{F(x) - F(a) - b(x - a)}{x - a} = \varepsilon(x)
$$

Since $\varepsilon(x) \to 0$ as $x \to a$, it follows that:

$$
\lim_{x \to a} \frac{F(x) - F(a) - b(x - a)}{x - a} = 0
$$

which completes the necessity part.

Sufficiency: If the limit holds, then F is differentiable at a with $F'(a) = b$

Conversely, assume:

$$
\lim_{x \to a} \frac{F(x) - F(a) - b(x - a)}{x - a} = 0
$$

Define:

$$
\varepsilon(x) = \frac{F(x) - F(a) - b(x - a)}{x - a}
$$

so that $\varepsilon(x) \to 0$ as $x \to a$.

Rearranged, this yields:

$$
F(x) - F(a) = b(x - a) + \varepsilon(x)(x - a)
$$

Dividing both sides by $(x - a)$:

$$
\frac{F(x) - F(a)}{x - a} = b + \varepsilon(x)
$$

Taking the limit as $x \to a$, since $\varepsilon(x) \to 0$, we get:

$$
\lim_{x \to a} \frac{F(x) - F(a)}{x - a} = b
$$

Hence, F is differentiable at a with derivative b.

Implications and Applications

Linear Approximation and Differentiability

The theorem essentially states that differentiability at a point is equivalent to the existence of a good linear approximation near that point. The function F can be approximated by its tangent line $L(x) = F(a) + b(x - a)$, with the error term $F(x) - F(a) - b(x - a)$ vanishing faster than $(x - a)$.

This understanding is fundamental in numerical analysis, optimization, and approximation theory. It allows us to approximate complex functions locally with linear functions, simplifying analysis and computations.

Connection to the Definition of the Derivative

The classic definition involves the limit of the difference quotient:

$$
F'(a) = \lim_{x \to a} \frac{F(x) - F(a)}{x - a}
$$

The theorem refines this by considering the remainder term $F(x) - F(a) - b(x - a)$. If this remainder is negligible compared to $(x - a)$, the function is differentiable with the derivative b.

Practical Examples

- Polynomial functions: They are differentiable everywhere, and the theorem confirms that the difference between the polynomial and its tangent line at a point vanishes faster than $(x - a)$.
- Absolute value function: Not differentiable at $x=0$, as the difference $F(x) - F(0) - 0 \cdot (x - 0)$ does not behave as $o(x)$.
- Exponential and trigonometric functions: These are differentiable everywhere, and the theorem applies straightforwardly to confirm their differentiability.

Summary

The core idea of the theorem is that the differentiability of F at a point a with derivative b is equivalent to the fact that:

$$
F(x) - F(a) - b(x - a) = o(x - a) \quad \text{as} \quad x \to a
$$

or, equivalently,

$$
\lim_{x \to a} \frac{F(x) - F(a) - b(x - a)}{x - a} = 0
$$

This characterization provides a powerful and intuitive criterion for differentiability, linking the limit definition of the derivative to the approximation of the function by its tangent line.

Conclusion

Proving that a function F is differentiable at a point a with derivative b involves understanding the behavior of the difference $F(x) - F(a) - b(x - a)$. The key equivalence states that this difference must be negligible compared to \((x -

Frequently Asked Questions

What is the necessary and sufficient condition for a function F to be differentiable at a point a with derivative F'(a) = b?

F is differentiable at a with F'(a) = b if and only if the limit as x approaches a of (F(x) - F(a) - b(x - a)) divided by (x - a) equals zero.

How does the limit definition relate to the differentiability of a function at a point?

The function F is differentiable at a if the limit of (F(x) - F(a) - b(x - a)) / (x - a) as x approaches a is zero, indicating the linear approximation with slope b accurately models F near a.

What role does the condition F(x) - F(a) - b(x - a) $\to$ 0 as x $\to$ a play in differentiability?

This condition ensures that the difference between the actual function and its tangent line at a vanishes faster than (x - a), confirming the existence of a derivative b at that point.

Can the converse be used to prove differentiability of a function at a point?

Yes, if the limit of (F(x) - F(a) - b(x - a)) / (x - a) as x approaches a is zero, then F is differentiable at a with derivative F'(a) = b.

Why is the expression F(x) - F(a) - b(x - a) important in the proof of differentiability?

Because it measures how well the linear approximation b(x - a) matches the function near a; its vanishing rate determines whether F is differentiable at a.

How does the limit condition ensure the differentiability criterion is both necessary and sufficient?

The limit condition guarantees that the difference quotient approaches zero, ensuring the existence of a derivative (necessity), and if the limit holds, the function can be locally approximated by a linear function (sufficiency).

Additional Resources

Prove That A Function F Is Differentiable At X = A With F'(A) = B, BeR, If And Only If F(x) - F(A) - B(x - A)

Understanding how to prove that a function F is differentiable at a point A with derivative F'(A) = B is fundamental in advanced calculus and real analysis. This criteria not only helps in establishing differentiability but also provides insight into the behavior of functions near specific points. The statement that F is differentiable at A with derivative B if and only if

$$ \lim_{x \to A} \frac{F(x) - F(A) - B(x - A)}{|x - A|} = 0 $$

serves as a cornerstone in the theory of derivatives. In this guide, we will explore this concept thoroughly, breaking down the proof into understandable segments, providing intuition, and illustrating key ideas with examples.

Introduction: The Significance of Differentiability and Its Characterizations

Differentiability at a point is a vital property in calculus, signifying that a function behaves locally like a linear function around that point. When a function is differentiable at a point A, it admits a linear approximation near A, which precisely embodies the derivative F'(A).

The classical definition states:

> A function F is differentiable at A if the limit

> $$ \lim_{x \to A} \frac{F(x) - F(A)}{x - A} $$

> exists and is finite, and this limit is called the derivative, denoted by F'(A).

However, an equivalent, more nuanced characterization involves examining the behavior of the difference $F(x) - F(A) - B(x - A)$ as $x \to A$. This form provides a stronger, more analytic condition that directly links differentiability to the concept of a "little-o" term, which vanishes faster than $|x - A|$.

The Core Theorem: Statement and Intuition

The Theorem

> Let F be a function defined on an open interval containing A, and let B be a real number. Then, F is differentiable at A with F'(A) = B if and only if

> $$ \lim_{x \to A} \frac{F(x) - F(A) - B(x - A)}{|x - A|} = 0. $$

Intuitive Explanation

- The expression $F(x) - F(A) - B(x - A)$ measures the deviation of the function from its tangent line at A.
- Dividing by $|x - A|$ normalizes this deviation, essentially examining the average difference per unit length as x approaches A.
- If the limit of this ratio is zero, it indicates that the deviation becomes negligible compared to $|x - A|$, meaning F is well-approximated by the linear function $F(A) + B(x - A)$ near A.
- Conversely, if F is differentiable at A with derivative B, then this deviation must vanish faster than $|x - A|$, ensuring the limit is zero.

Formal Proof of the "If and Only If" Statement

Let's split the proof into two parts:

Part 1: If F is differentiable at A with F'(A) = B, then the limit is zero.

Part 2: If the limit is zero, then F is differentiable at A with F'(A) = B.

Part 1: Differentiability Implies the Limit is Zero

Assumption: F is differentiable at A and $F'(A) = B$.

Objective: Show that

$$ \lim_{x \to A} \frac{F(x) - F(A) - B(x - A)}{|x - A|} = 0. $$

Proof Sketch:

1. Differentiability definition: By the limit definition,

$$ \lim_{x \to A} \frac{F(x) - F(A)}{x - A} = B. $$

2. Express the difference: Rewrite $F(x) - F(A)$ as

$$ (F(x) - F(A)) = B(x - A) + \varepsilon(x), $$

where

$$ \varepsilon(x) = F(x) - F(A) - B(x - A). $$

3. Behavior of $\varepsilon(x)$: Because the limit exists,

$$ \lim_{x \to A} \frac{\varepsilon(x)}{x - A} = 0, $$

which implies $\varepsilon(x) = o(x - A)$.

4. Limit of the ratio: Now, examine

$$ \frac{F(x) - F(A) - B(x - A)}{|x - A|} = \frac{\varepsilon(x)}{|x - A|}. $$

Since $\varepsilon(x) = o(x - A)$, the ratio tends to zero as $x \to A$.

Conclusion: The limit

$$ \lim_{x \to A} \frac{F(x) - F(A) - B(x - A)}{|x - A|} = 0. $$

Part 2: The Limit Zero Implies Differentiability with Derivative B

Assumption:

$$ \lim_{x \to A} \frac{F(x) - F(A) - B(x - A)}{|x - A|} = 0. $$

Objective: Show that F is differentiable at A with $F'(A) = B$.

Proof Sketch:

1. Define $\varepsilon(x)$: As before,

$$ \varepsilon(x) = F(x) - F(A) - B(x - A). $$

2. From the limit assumption:

$$ \lim_{x \to A} \frac{\varepsilon(x)}{|x - A|} = 0, $$

which implies

$$ \varepsilon(x) = o(|x - A|). $$

3. Establish the difference quotient:

$$ \frac{F(x) - F(A)}{x - A} = \frac{B(x - A) + \varepsilon(x)}{x - A} = B + \frac{\varepsilon(x)}{x - A}. $$

4. Limit of the difference quotient:

Since $\varepsilon(x) = o(|x - A|)$, the second term tends to zero as $x \to A$.

Therefore,

$$ \lim_{x \to A} \frac{F(x) - F(A)}{x - A} = B. $$

5. Conclusion:

The limit exists and equals B, confirming F is differentiable at A with derivative $F'(A) = B$.

Practical Implications and Applications

1. Verifying Differentiability

The theorem provides a powerful criterion for confirming whether a function is differentiable at a point, especially when the derivative is known or guessed. Instead of directly computing the limit of the difference quotient, one can analyze the behavior of $F(x) - F(A) - B(x - A)$ relative to $|x - A|$.

2. Constructing Derivatives

In cases where the derivative exists, this criterion helps in constructing the derivative by examining the limiting behavior of the normalized difference. It also assists in understanding the rate at which the function approaches its tangent line.

3. Handling Non-Differentiable Functions

If the limit is not zero, the function is not differentiable at the point with the proposed derivative B. This helps identify points of non-differentiability or confirm differentiability when the limit is zero.

Examples to Illustrate the Criterion

Example 1: Differentiable Function

Let $F(x) = x^2$, and evaluate at $A = 1$.

- $F(1) = 1$,
- $F'(1) = 2$,
- Choose $B = 2$.

Compute

$$\frac{F(x) - F(1) - 2(x - 1)}{|x - 1|} = \frac{x^2 - 1 - 2(x - 1)}{|x - 1|}.$$

Simplify numerator:

$$x^2 - 1 - 2x + 2 = x^2 - 2x + 1 = (x - 1)^2.$$

Thus,

$$\frac{(x - 1)^2}{|x - 1|} = |x - 1|.$$

As $x \to 1$, $|x - 1| \to 0$. Therefore,

$$\lim_{x \to 1} \frac{F(x) - F(1) - 2(x - 1)}{|x - 1|} = 0.$$

Hence, the criterion confirms the differentiability of F at 1 with derivative 2.

Example

Prove That A Function F Is Differentiable At X A With F A B Ber If And Only If F X F A B X A

Prove That A Function F Is Differentiable At X A With F A B Ber If And Only If F X F A B X A

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