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Understanding the Problem

In this article, we delve into a comprehensive analysis of the matrix \(A\) defined as \(A = [1\, 1\, 0\, 1\, 1]\). The problem involves two main tasks:
1. Finding the singular values of matrix \(A\).
2. Determining a unit vector \(X\) such that \(AX = \lambda X\), where \(\lambda\) is an eigenvalue associated with \(A\).

This problem touches upon fundamental concepts in linear algebra, specifically singular value decomposition (SVD) and eigenvalues/eigenvectors, which are instrumental in numerous scientific and engineering applications such as data compression, principal component analysis, and systems stability.

Part A: Finding the Singular Values of A

What Are Singular Values?

Singular values are non-negative real numbers that provide insights into the properties of a matrix. For a matrix \(A\), the singular values are the square roots of the eigenvalues of the matrix \(A^T A\). They are obtained through the Singular Value Decomposition (SVD), which factors \(A\) into the product of three matrices:
\[
A = U \Sigma V^T
\]
where:
- \(U\) and \(V\) are orthogonal matrices,
- \(\Sigma\) is a diagonal matrix containing the singular values.

Representing the Matrix \(A\)

Given the matrix:
\[
A = [1\, 1\, 0\, 1\, 1]
\]
which can be written as a \(1 \times 5\) matrix (a row vector):

$$
A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \end{bmatrix}
$$
This is a row vector, and the calculation of singular values simplifies to analyzing this vector's properties.

Calculating $A^T A$

To find the singular values, we compute:
$$
A^T A
$$
which results in a 5×5 matrix:
$$
A^T A = \begin{bmatrix}
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
\end{bmatrix}
$$

However, because A is a single row vector, the singular values are simply the Euclidean norm of A and its zero values.

Key Point: For a $1 \times n$ matrix, the singular values are:
- $\sigma_1 = \|A\|_2 = \sqrt{1^2 + 1^2 + 0^2 + 1^2 + 1^2} = \sqrt{1 + 1 + 0 + 1 + 1} = \sqrt{4} = 2$.
- All other singular values are zero because the rank of A is 1.

Result:
$$
\boxed{
\text{Singular Values of } A: \quad \sigma_1 = 2, \quad \sigma_2 = 0, \quad \sigma_3=0, \quad \dots
}
$$
since the matrix is rank 1, only the first singular value is non-zero.

Part B: Finding A Unit Vector X for Which $AX = \lambda X$

Eigenvalues and Eigenvectors of A

The second part of the problem involves identifying a unit vector X such that the matrix-vector product yields a scalar multiple of X:

$$
AX = \lambda X
$$
This is the definition of an eigenvector X associated with eigenvalue λ.

However, since A is a 1×5 matrix, it cannot have eigenvalues and eigenvectors in the traditional sense because eigenvalues are defined for square matrices.

But, in the context of the problem, it might be intended that we consider the matrix $A^T A$, which is a 5×5 symmetric matrix and does have eigenvalues and eigenvectors.

Eigenvalues of $A^T A$

Recall that the singular values of A are the square roots of the eigenvalues of $A^T A$. Since we have:
$$
\sigma_1 = 2
$$
then the eigenvalue corresponding to the largest singular value is:
$$
\lambda_1 = \sigma_1^2 = 4
$$

The eigenvalues of $A^T A$ are thus:
- $\lambda_1 = 4$,
- and the remaining eigenvalues are zero (due to the rank of A).

Eigenvector Corresponding to $\lambda_1 = 4$

To find the eigenvector X associated with the eigenvalue $\lambda_1=4$, we need to solve:
$$
A^T A X = 4 X
$$
or equivalently:
$$
A X = \sigma_1 u
$$
where u is the left singular vector corresponding to σ_1.

Since A is a row vector, the singular vector u in $\mathbb{R}^1$ is simply 1, and the right singular vector v in $\mathbb{R}^5$ is proportional to A^T.

Calculating X:
Given that the singular value $\sigma_1 = 2$, the corresponding right singular vector X can be obtained as:

$$
X = \frac{A^T}{\|A\|_2} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}
$$

Verification:

$$
A X = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} (1 \times 1 + 1 \times 1 + 0 \times 0 + 1 \times 1 + 1 \times 1) = \frac{1}{2} (1 + 1 + 0 + 1 + 1) = \frac{1}{2} \times 4 = 2
$$

which aligns with the singular value.

The vector:

$$
X = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}
$$

is a unit vector because:

$$
\|X\|_2 = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 0^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{1}{4}} = \sqrt{1} = 1
$$

Summary of Key Results

- **Singular Values of A:** $\sigma_1 = 2$, and all others are zero.

- **Unit Vector X:** $\boxed{ X = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} }$, which satisfies $AX = 2$, the dominant singular value.

Practical Applications

Understanding singular values and vectors is crucial in numerous fields:
- Data Compression: SVD helps reduce the dimensionality of datasets while preserving essential information.
- Machine Learning: Principal Component Analysis (PCA) relies on eigenvalues and eigenvectors derived from data matrices.
- Signal Processing: SVD aids in noise reduction and signal filtering.
- Control Systems: Eigenvalues determine system stability, while singular values assess system robustness.

Conclusion

In conclusion, the analysis of matrix \(A = [1\, 1\, 0\, 1\, 1]\) reveals that its singular value decomposition is straightforward due to its structure as a row vector. The primary singular value is

Frequently Asked Questions

How do you compute the singular values of a matrix A?

To find the singular values of matrix A, compute the eigenvalues of the matrix A^tA, then take the square roots of these eigenvalues. The singular values are always non-negative and arranged in descending order.

Given the matrix A = [[1, 1], [0, 1], [1, 1]], how do you find its singular values?

First, compute A^tA = [[2, 2], [2, 3]]. Then find the eigenvalues of A^tA, which are 0.381966 and 4.618034. The singular values are the square roots of these eigenvalues, approximately 0.618 and 2.149.

What approach is used to find a unit vector X such that Ax = λx for a given matrix A?

This involves solving the eigenvalue problem Ax = λx for A. The eigenvector X corresponding to eigenvalue λ is found by solving the homogeneous system (A - λI)x = 0, then normalizing the vector to make it a unit vector.

In the context of Question 2, how can you find a unit vector X satisfying Ax'?

Identify the eigenvector corresponding to the singular value (which acts as the scalar λ) and normalize it to obtain the unit vector X. This vector satisfies the relation Ax = λx.

Why are singular values important in understanding the properties of matrix A?

Singular values measure the magnitude of the action of A on vectors, indicating how much A stretches or compresses vectors. They are essential in applications like data compression, noise reduction, and solving least squares problems.

Can the singular values of matrix A be zero, and what does that imply?

Yes, zero singular values indicate that A is rank-deficient, meaning it maps some non-zero vectors to the zero vector. This implies that A does not have full rank and is not invertible.

Additional Resources

Deep Dive into Singular Values and Special Vectors of a Matrix: Analyzing A = $\begin{bmatrix}1 & 1 \\ 0 & 1 \\ 1 & 1\end{bmatrix}$

Introduction to the Problem

In the realm of linear algebra, matrices serve as fundamental building blocks for understanding linear transformations, systems of equations, and various applications in engineering, physics, and computer science. When analyzing a matrix, two vital concepts often come into play: singular values and singular vectors. These concepts are essential for understanding the geometric and algebraic properties of matrices, especially in contexts like data compression, noise reduction, and principal component analysis.

Given the matrix:

$$
A = \begin{bmatrix}
1 & 1 \\
0 & 1 \\
1 & 1
\end{bmatrix}
$$

we are tasked with:

- Part (a): Finding the singular values of A.
- Part (b): Finding a unit vector X such that $A X$ has some specified property (commonly, maximizing $ \|A X\| $ or in the context of singular vectors, identifying the right singular vector corresponding to the largest singular value).

This detailed exploration aims to not only compute the answers but also to provide a comprehensive understanding of the underlying concepts, methods, and implications.

Understanding Singular Values and Singular Vectors

Before diving into calculations, it's crucial to grasp what singular values and vectors represent:

- Singular Values: The non-negative square roots of the eigenvalues of $A^T A$ (or $A A^T$). They measure the "stretching" effect of the matrix on vectors—how much the matrix can amplify or diminish the length of vectors in its domain.

- Singular Vectors: Corresponding to these singular values, the right singular vectors (X) are eigenvectors of $A^T A$, and the left singular vectors (Y) are eigenvectors of $A A^T$. They form orthonormal bases for the domain and codomain spaces, respectively.

The singular value decomposition (SVD) of A expresses A as:

$$
A = U \Sigma V^T
$$

where:

- U contains the left singular vectors,
- Σ is a diagonal matrix with singular values,
- V contains the right singular vectors.

Part (a): Computing the Singular Values of A

Step 1: Compute $A^T A$

Given

$$
A = \begin{bmatrix}
1 & 1 \\
0 & 1 \\
1 & 1
\end{bmatrix}
$$

then its transpose:

$$
A^T = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}
$$

Now,

$$
A^T A =
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
1 & 1
\end{bmatrix}
$$

Compute element-wise:

- Entry (1,1):
$$
(1)(1) + (0)(0) + (1)(1) = 1 + 0 + 1 = 2
$$
- Entry (1,2):
$$
(1)(1) + (0)(1) + (1)(1) = 1 + 0 + 1 = 2
$$
- Entry (2,1):
$$
(1)(1) + (1)(0) + (1)(1) = 1 + 0 + 1 = 2
$$
- Entry (2,2):
$$
(1)(1) + (1)(1) + (1)(1) = 1 + 1 + 1 = 3
$$

Thus,
$$
A^T A = \begin{bmatrix}
2 & 2 \\
2 & 3
\end{bmatrix}
$$

Step 2: Find eigenvalues of $A^T A$

The eigenvalues λ satisfy:
$$
\det(A^T A - \lambda I) = 0
$$
$$
\det \begin{bmatrix} 2 - \lambda & 2 \\ 2 & 3 - \lambda \end{bmatrix} = 0
$$

Compute the determinant:
$$
(2 - \lambda)(3 - \lambda) - (2)(2) = 0
$$
$$
(2 \times 3) - 2 \lambda - 3 \lambda + \lambda^2 - 4 = 0
$$
$$
6 - 5 \lambda + \lambda^2 - 4 = 0
$$
$$
(\lambda^2 - 5 \lambda + 2) = 0
$$

Step 3: Solve the quadratic

$$
\lambda^2 - 5 \lambda + 2 = 0
$$

Using quadratic formula:

$$
\lambda = \frac{5 \pm \sqrt{25 - 8}}{2} = \frac{5 \pm \sqrt{17}}{2}
$$

Thus, the eigenvalues are:

$$
\boxed{
\lambda_1 = \frac{5 + \sqrt{17}}{2} \quad \text{and} \quad \lambda_2 = \frac{5 - \sqrt{17}}{2}
}
$$

Step 4: Compute the singular values

Recall that singular values are the square roots of the eigenvalues of $A^T A$:

$$
\sigma_i = \sqrt{\lambda_i}
$$

Hence,

$$
\sigma_1 = \sqrt{\frac{5 + \sqrt{17}}{2}}
$$

$$
\sigma_2 = \sqrt{\frac{5 - \sqrt{17}}{2}}
$$

Numerical Approximations:

$$
\sqrt{17} \approx 4.1231
$$

So,

$$
\lambda_1 \approx \frac{5 + 4.1231}{2} \approx \frac{9.1231}{2} \approx 4.56155
$$

$$
\lambda_2 \approx \frac{5 - 4.1231}{2} \approx \frac{0.8769}{2} \approx 0.43845
$$

Corresponding singular values:

$$
\sigma_1 \approx \sqrt{4.56155} \approx 2.136
$$
$$
\sigma_2 \approx \sqrt{0.43845} \approx 0.662
$$

Summary of Part (a):

$$
\boxed{
\text{Singular values of }A: \quad \sigma_1 \approx 2.136, \quad \sigma_2 \approx 0.662
}
$$

These singular values quantify the maximum and minimum "stretching" effect of the matrix A. The larger singular value indicates the direction in which the matrix stretches vectors the most.

Part (b): Finding a Unit Vector X such that $A X$ is maximized

This part often corresponds to finding the right singular vector associated with the largest singular value, σ_1.

Objective:

Identify a unit vector X (with $\|X\| = 1$) such that the norm $\|A X\|$ is maximized, which occurs when X is the right singular vector corresponding to σ_1.

Step 1: Find the eigenvector of $A^T A$ associated with λ_1

Recall the eigenvalue problem:
$$
A^T A \mathbf{x} = \lambda \mathbf{x}
$$
where $\lambda = \lambda_1 = \frac{5 + \sqrt{17}}{2}$.

Using the matrix:
$$
A^T A = \begin{bmatrix}
2 & 2 \\

2 & 3
\end{bmatrix}
\]

and the eigenvalue \(\lambda_1\), solve:
\[
\left(A^T A - \lambda_1 I \right) \mathbf{x} = 0
\]

Calculate:
\[
A^T A - \lambda_1 I =
\begin{bmatrix}
2 - \lambda_1 & 2 \\
2 & 3 - \lambda_1
\end{bmatrix}
\]

Substitute \(\lambda_1 \approx 4.56155\):

\[
2 - 4.56155 \approx -2.56155
\]
\[
3 - 4.56155 \approx -1.56155
\]

So,
\[
\begin{bmatrix}
-2.56155 & 2 \\
2 & -1.56155

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