Second Order ODEs With Constant Coefficients Solve The Following Initial Value Problem: Y" - 4y' + 4y

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Introduction

Second order ordinary differential equations (ODEs) with constant coefficients are a fundamental part of differential equations, often encountered in physics, engineering, and applied mathematics. These equations describe systems where the rate of change depends on the current state and its derivatives, with the coefficients of the derivatives being constants. Solving such equations involves characteristic equations, roots, and, when applicable, initial conditions that specify the solution uniquely.

This article aims to provide a comprehensive guide to solving second order ODEs with constant coefficients by working through a specific initial value problem: Y" - 4y' + 4y. We will explore the theory behind these equations, methods for solving them, and detailed steps to obtain the solution, including handling initial conditions.

Understanding the Differential Equation

The given differential equation is:

$$ Y'' - 4Y' + 4Y = 0 $$

where:
- Y'' is the second derivative of y with respect to x,
- Y' is the first derivative,
- Y is the function of x.

This is a homogeneous linear second order differential equation with constant coefficients because the coefficients (1, -4, 4) are constants.

Step 1: Write the Characteristic Equation

To solve such equations, we typically assume solutions of the form $y = e^{rx}$, where r is a constant to be determined. Substituting into the differential equation yields the characteristic equation:

$$ r^2 - 4r + 4 = 0 $$

This quadratic equation encapsulates the behavior of the differential equation.

Step 2: Solve the Characteristic Equation

The quadratic equation:

$$ r^2 - 4r + 4 = 0 $$

can be factored or solved using the quadratic formula:

$$ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} $$

where $a=1$, $b=-4$, $c=4$.

Calculating discriminant:

$$ \Delta = (-4)^2 - 4 \times 1 \times 4 = 16 - 16 = 0 $$

Since the discriminant is zero, the characteristic roots are real and equal:

$$ r = \frac{4}{2} = 2 $$

Thus, both roots are $r = 2$.

Step 3: Write the General Solution

For repeated roots in the characteristic equation, the general solution to the differential equation is:

$$ y(x) = (A + Bx) e^{rx} $$

where A and B are arbitrary constants determined by initial conditions, and $r = 2$.

Therefore, the general solution is:

$$ y(x) = (A + Bx) e^{2x} $$

Step 4: Apply Initial Conditions

Suppose the initial conditions are given as:

$$
\begin{cases}
y(0) = y_0 \\
y'(0) = y_1
\end{cases}
$$

to find constants A and B.

Step 5: Find $y'(x)$

Differentiate $y(x)$:

$$
y(x) = (A + Bx) e^{2x}
$$

Using the product rule:

$$
y'(x) = \frac{d}{dx} [(A + Bx) e^{2x}] = (A + Bx) \times 2e^{2x} + B e^{2x} = e^{2x} [2(A + Bx) + B]
$$

Simplify:

$$
y'(x) = e^{2x} [2A + 2B x + B]
$$

Step 6: Implement Initial Conditions

Evaluating at $x=0$:

$$
y(0) = (A + B \times 0) e^{0} = A \times 1 = A
$$

$$
y'(0) = e^{0} [2A + 2B \times 0 + B] = 1 \times (2A + B) = 2A + B
$$

Given $y(0) = y_0$, $y'(0) = y_1$:

$$
A = y_0
$$

$$
2A + B = y_1 \Rightarrow 2 y_0 + B = y_1 \Rightarrow B = y_1 - 2 y_0
$$

Final Solution with Initial Conditions

Substituting back:

$$
\boxed{
y(x) = \left(y_0 + (y_1 - 2 y_0) x \right) e^{2x}
}
$$

This is the unique solution to the initial value problem.

Additional Insights and Applications

Homogeneous vs. Non-Homogeneous Equations

The differential equation we solved is homogeneous, meaning the right-hand side is zero. Non-homogeneous equations involve a non-zero term (forcing function) and require particular solutions, often found via methods such as undetermined coefficients or variation of parameters.

Physical Interpretation

Such equations model systems like:

- Damped harmonic oscillators,
- RC or RLC circuits,
- Mechanical vibrations.

The repeated root indicates a critically damped system in physics, where the system returns to equilibrium as quickly as possible without oscillating.

General Strategy for Solving Second Order ODEs with Constant Coefficients

1. Write the characteristic equation based on the differential equation.
2. Solve the quadratic to find roots.
3. Determine the form of the general solution:
- Distinct real roots: $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
- Repeated roots: $y = (A + Bx) e^{r x}$
- Complex roots: $y = e^{\alpha x} (C \cos \beta x + D \sin \beta x)$
4. Apply initial conditions to solve for arbitrary constants.
5. Interpret the solution in context.

Common Pitfalls and Tips

- Incorrectly solving the characteristic equation can lead to wrong solutions.

- For repeated roots, remember the form involves an x term.
- Always check initial conditions carefully to determine constants.
- In cases of complex roots, convert to sines and cosines.
- Ensure the solution is simplified and constants are correctly substituted.

Conclusion

Solving second order ODEs with constant coefficients is a systematic process that hinges on solving the characteristic equation and applying initial conditions. The example $y'' - 4 y' + 4 y = 0$ demonstrates how repeated roots influence the form of the general solution, leading to solutions involving both exponential and polynomial factors.

Understanding these methods equips you with the tools to approach a wide range of differential equations encountered in physics, engineering, and beyond. Whether modeling oscillations, electrical circuits, or mechanical systems, mastering the solution techniques for second order ODEs with constant coefficients is essential for advanced mathematical analysis and practical problem-solving.

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Frequently Asked Questions

What is the general form of a second-order linear differential equation with constant coefficients?

It is generally written as aY'' + bY' + cY = 0, where a, b, and c are constants.

How do you find the characteristic equation for the differential equation Y'' - 4Y' + 4Y = 0?

Replace Y'' with r^2, Y' with r, and Y with 1 to get the characteristic equation: r^2 - 4r + 4 = 0.

What is the characteristic equation for the differential equation Y'' - 4Y' + 4Y = 0?

The characteristic equation is r^2 - 4r + 4 = 0.

How do you solve the characteristic equation r^2 - 4r + 4 = 0?

Factor it as (r - 2)^2 = 0, giving a repeated root r = 2.

What is the general solution for the differential equation with repeated roots r = 2?

The general solution is Y(t) = (C1 + C2 t) e^{2t}.

Given the initial conditions Y(0) = Y0 and Y'(0) = Y1, how do you find the particular solution?

Substitute t=0 into the general solution and its derivative, then solve for C1 and C2 using the initial conditions.

What is the importance of the initial conditions in solving second-order ODEs?

They allow you to determine the specific values of the constants C1 and C2, leading to a unique solution.

Can you provide an example initial value problem for this differential equation?

Sure, for example: Y'' - 4Y' + 4Y = 0, with Y(0) = 1 and Y'(0) = 0.

How do repeated roots affect the form of the solution to second-order linear ODEs?

Repeated roots lead to solutions involving a polynomial multiplied by the exponential, such as t e^{rt}, to account for multiplicity.

Additional Resources

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Introduction

In the realm of differential equations, second order ODEs with constant coefficients occupy a pivotal position due to their broad applicability across physics, engineering, and applied mathematics. These equations often serve as the foundational models for oscillatory systems, mechanical vibrations, electrical circuits, and more. Among them, the general form:

$$ y'' + a y' + b y = 0 $$

(where a, b are constants) encapsulates a wide class of problems. Their solutions are well-understood thanks to the characteristic equation method, which transforms the differential equation into an algebraic problem.

This article undertakes a meticulous investigation of such an equation, specifically focusing on solving the initial value problem (IVP):

$$ y'' - 4 y' + 4 y = 0 $$

with given initial conditions, typically $y(0) = y_0$ and $y'(0) = y_1$. The goal is to explore the solution process comprehensively, elucidate the mathematical structure, and analyze the implications of the solutions in various contexts.

Fundamental Concepts and Mathematical Framework

The Form of the Differential Equation

The given differential equation is:

$$ y'' - 4 y' + 4 y = 0 $$

which is linear, homogeneous, and has constant coefficients. Its solution hinges on the roots of the characteristic equation derived from the differential operator.

Characteristic Equation

Transforming the differential equation into an algebraic form involves substituting $y = e^{r t}$, yielding:

$$ r^2 - 4 r + 4 = 0 $$

This quadratic polynomial, known as the characteristic equation, encapsulates the behavior of the differential equation’s solutions.

Solving the Characteristic Equation

Roots of the Characteristic Polynomial

The roots of the quadratic are given by:

$$ r = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 4}}{2} $$

Calculating discriminant:

$$ \Delta = 16 - 16 = 0 $$

Since the discriminant is zero, the roots are real and equal:

$$ r = \frac{4}{2} = 2 $$

with multiplicity two.

Implication of Repeated Roots

Repeated roots indicate that the general solution must include a polynomial term multiplied by the exponential function to account for multiplicity. Specifically:

$$ y(t) = (A + B t) e^{r t} $$

where A and B are arbitrary constants determined by initial conditions.

General Solution to the Differential Equation

Given the roots, the general solution is:

$$ y(t) = (A + B t) e^{2 t} $$

This form encompasses all solutions to the differential equation, with parameters A, B to be specified by initial conditions.

Applying Initial Conditions

Suppose the initial conditions are:

$$ y(0) = y_0 $$
$$ y'(0) = y_1 $$

To find A and B, compute $y(t)$ and $y'(t)$:

- $y(t) = (A + B t) e^{2 t}$

- $y'(t) = \frac{d}{dt} [(A + B t) e^{2 t}]$

Applying the product rule:

$$ y'(t) = B e^{2 t} + (A + B t) \times 2 e^{2 t} = \left(B + 2 A + 2 B t \right) e^{2 t} $$

At $t=0$:

$$y(0) = A e^{0} = A = y_0$$

$$y'(0) = \left(B + 2 A \right) e^{0} = B + 2 A = y_1$$

From these, solve for A, B:

$$A = y_0$$

$$B = y_1 - 2 y_0$$

Thus, the particular solution becomes:

$$y(t) = \left(y_0 + (y_1 - 2 y_0) t \right) e^{2 t}$$

Significance and Interpretation of the Solution

Behavior of the Solution

The exponential component $e^{2 t}$ indicates growth or decay depending on context, but here, with a positive exponent, the solution exhibits exponential growth as $t \to \infty$. The linear term in t modulates this growth, and the initial conditions dictate the specific trajectory.

Stability Analysis

In systems modeled by such equations, the repeated root at $r=2$ reflects a so-called degenerate case, often associated with critical damping in mechanical systems. This is crucial when analyzing systems like the damped harmonic oscillator, where the damping coefficient matches the natural frequency, leading to a solution involving polynomial growth multiplied by exponential decay or growth.

Deep Dive: Physical and Mathematical Contexts

Mechanical Vibrations

In mechanical systems, equations of the form:

$$y'' + 2 \zeta \omega_n y' + \omega_n^2 y = 0$$

describe damped oscillations, where ζ is the damping ratio and ω_n the natural frequency. When $2 \zeta \omega_n = 4$, the characteristic roots coincide, indicating critical damping. The solution derived earlier corresponds to this critical damping case, where the system returns to equilibrium as quickly as possible without oscillating.

Electrical Circuits

Similarly, in RLC circuits, the differential equations governing current or voltage often reduce to second order linear equations with constant coefficients. The nature of roots influences whether the circuit exhibits oscillatory behavior or returns to steady state monotonically.

Extension: Non-Homogeneous Equations and Variations

While the focus here is on the homogeneous equation, real-world problems often involve non-homogeneous terms. For example:

$$ y'' - 4 y' + 4 y = g(t) $$

where $g(t)$ is a forcing function. The solution methodology extends via the method of undetermined coefficients or variation of parameters, adding particular solutions to the homogeneous solution.

Numerical Methods and Practical Approaches

In many practical scenarios, exact solutions, while insightful, may be insufficient or infeasible due to complex initial conditions or non-linearities. Numerical methods such as Runge-Kutta or finite difference schemes are employed to approximate solutions, especially for larger systems or when dealing with non-constant coefficients.

Summary and Conclusions

The differential equation:

$$ y'' - 4 y' + 4 y = 0 $$

exemplifies the classical second order linear differential equation with constant coefficients. Its solution process underscores the importance of the characteristic equation, the role of root multiplicity, and the form of the general solution.

Key takeaways include:

- The characteristic equation's roots determine the solution structure.
- Repeated roots lead to polynomial-exponential solutions.
- Initial conditions uniquely specify the arbitrary constants.
- The solutions' behavior reflects physical properties such as damping and stability.

Understanding these solutions enhances our capacity to model, analyze, and predict the behavior of complex systems in engineering, physics, and applied sciences. The methodology exemplified here forms the backbone of differential equations analysis and remains a vital tool in scientific inquiry.

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