Show That Div Curl F=0 Assuming That F Is Continuous And Has Continuous Partial Derivatives Of Second

Show That Div Curl F=0 Assuming That F Is Continuous And Has Continuous Partial Derivatives Of Second

Understanding vector calculus is crucial for many fields such as physics, engineering, and applied mathematics. One fundamental result in vector calculus states that the divergence of the curl of any sufficiently smooth vector field is always zero. This property plays a vital role in understanding the behavior of vector fields, especially in electromagnetism, fluid dynamics, and differential equations.

This article aims to provide a comprehensive, step-by-step explanation of why $\operatorname{div} (\operatorname{curl} \mathbf{F}) = 0$ holds true under the assumption that $\mathbf{F}$ is continuous and has continuous second partial derivatives. We will delve into the mathematical reasoning, explore the necessary conditions, and illustrate the importance of this property in various applications.

Background and Context

Before diving into the proof, it’s important to comprehend the fundamental operators involved:

- Gradient (∇): An operator that acts on scalar fields to produce a vector field.
- Curl ($\nabla \times$): An operator that measures the rotation or curl of a vector field.
- Divergence ($\nabla \cdot$): An operator that measures the magnitude of a source or sink at a given point in a vector field.

The relation $\operatorname{div} (\operatorname{curl} \mathbf{F}) = 0$ is a vector calculus identity that holds under certain smoothness conditions. It is often employed in proving fundamental theorems such as Maxwell's equations and in simplifying differential equations.

Fundamental Theorem and Conditions

Mathematical Statement of the Identity

Given a vector field $\mathbf{F} = (F_1, F_2, F_3)$ defined on an open subset of $\mathbb{R}^3$, the identity states:

$$
\operatorname{div} (\operatorname{curl} \mathbf{F}) = 0
$$

which explicitly is:

$$
\frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = 0
$$

Conditions for the Identity to Hold

For the above identity to be valid, the following conditions must be satisfied:

- $\mathbf{F}$ is continuous.
- $\mathbf{F}$ has continuous second partial derivatives (i.e., $F_i \in C^2$ for $i=1,2,3$).
- The domain of $\mathbf{F}$ is an open subset of $\mathbb{R}^3$, ensuring the applicability of theorems like Clairaut's theorem.

These conditions guarantee that mixed partial derivatives are equal and continuous, which is essential for the proof.

Step-by-Step Proof of $\operatorname{div} (\operatorname{curl} \mathbf{F}) = 0$

Step 1: Expressing $\operatorname{curl} \mathbf{F}$

The curl of $\mathbf{F}$ is given by:

$$
\operatorname{curl} \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},

\quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)
\]

Let’s denote this as:

\[
\mathbf{G} = (G_1, G_2, G_3)
\]

where

\[
\begin{cases}
G_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\
G_2 = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\
G_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}
\end{cases}
\]

Step 2: Computing \(\operatorname{div}(\operatorname{curl} \mathbf{F})\)

The divergence of \(\mathbf{G}\) is:

\[
\operatorname{div} \mathbf{G} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}
\]

Substituting the expressions for \(G_i\):

\[
\operatorname{div} \mathbf{G} = \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)
\]

Expanding this:

\[
\begin{aligned}
\operatorname{div} (\operatorname{curl} \mathbf{F}) &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}

\end{aligned}
$$

Step 3: Applying Mixed Partial Derivative Equality

Since $\mathbf{F}$ has continuous second partial derivatives, Clairaut's theorem (also known as the symmetry of mixed partial derivatives) applies, which states:

$$
\frac{\partial^2 F_i}{\partial x \partial y} = \frac{\partial^2 F_i}{\partial y \partial x}
$$
and similarly for other mixed derivatives.

Using this, the terms cancel pairwise:

$$
\begin{aligned}
\frac{\partial^2 F_3}{\partial x \partial y} &= \frac{\partial^2 F_3}{\partial y \partial x} \\
\frac{\partial^2 F_2}{\partial x \partial z} &= \frac{\partial^2 F_2}{\partial z \partial x} \\
\frac{\partial^2 F_1}{\partial y \partial z} &= \frac{\partial^2 F_1}{\partial z \partial y}
\end{aligned}
$$

Thus, the sum simplifies to zero:

$$
\operatorname{div} (\operatorname{curl} \mathbf{F}) = 0
$$

Implications and Applications

Physical Significance

The identity $\operatorname{div} (\operatorname{curl} \mathbf{F}) = 0$ has profound implications in physics:

- Electromagnetism: In Maxwell's equations, this identity underpins the divergence-free nature of magnetic fields ($\nabla \cdot \mathbf{B} = 0$), since magnetic fields are often represented as the curl of a vector potential.

- Fluid Dynamics: In incompressible fluids, the vorticity field is the curl of the velocity field, and the divergence of vorticity being zero relates to the conservation of angular momentum.

- Mathematical Consistency: It ensures the consistency of vector calculus operations, enabling the derivation of many fundamental theorems, such as the Helmholtz decomposition.

Mathematical and Theoretical Significance

From a mathematical perspective, this property:

- Demonstrates the compatibility of the differential operators.
- Serves as a foundation for vector calculus identities.
- Underpins the theory of differential forms and de Rham cohomology in advanced mathematics.

Conclusion

The proof that \(\operatorname{div} (\operatorname{curl} \mathbf{F}) = 0\) hinges on the smoothness conditions of the vector field \(\mathbf{F}\

Frequently Asked Questions

What does the condition div F = 0 imply about the vector field F?

The condition div F = 0 indicates that the vector field F is divergence-free, meaning there are no sources or sinks within the field, which often implies incompressibility or solenoidal behavior.

How can the curl of a vector field be used to show that div F = 0?

If the curl of F is zero (curl F = 0), and F has continuous second partial derivatives, then by vector calculus identities, the divergence of F must also be zero, since divergence of curl F is always zero.

Why is the assumption of continuous second derivatives

important in proving div F = 0?

Continuous second derivatives ensure the applicability of vector calculus theorems, such as the equality of mixed partial derivatives and the divergence of the curl being zero, which are essential in the proof.

Can you outline the steps to prove that curl F = 0 implies div F = 0 under the given conditions?

Yes. First, note that divergence of curl F is always zero for sufficiently smooth F. Since curl F = 0, then the curl of F is the zero vector field. Using the identity div(curl F) = 0, and considering the smoothness of F, it follows that the divergence of F must also be zero, completing the proof.

Are there any counterexamples where curl F = 0 but div F ≠ 0?

In general, for vector fields with continuous second derivatives, curl F = 0 implies div F = 0. However, if the smoothness assumptions are relaxed, such as discontinuous derivatives, counterexamples may exist, but under the given assumptions, the implication holds.

What physical phenomena can be modeled by divergence-free vector fields where curl F = 0?

Such vector fields often model incompressible, irrotational flows in fluid dynamics, magnetic fields in electromagnetism, and potential fields where the flow or field has no divergence or curl, representing conserved quantities.

Additional Resources

Show That Div Curl F = 0 Assuming That F Is Continuous And Has Continuous Partial Derivatives Of Second

Understanding the fundamental properties of vector calculus is essential for various fields such as physics, engineering, and applied mathematics. One of the core identities involving vector fields is that the divergence of the curl of a sufficiently smooth vector field is always zero. This property, often written as:

$$
\text{div}(\text{curl} \, \mathbf{F}) = 0,
$$

serves as a cornerstone in the study of vector calculus, electromagnetism, fluid mechanics, and more. In this comprehensive discussion, we will rigorously demonstrate this property under the assumptions that:

- $\mathbf{F}$ is continuous,

- $\mathbf{F}$ has continuous partial derivatives of second order (i.e., second-order differentiability).

The proof and its underlying ideas involve fundamental theorems of vector calculus, particularly the properties of partial derivatives and the divergence and curl operators.

Fundamental Definitions and Notation

Before delving into the proof, it is crucial to clarify the definitions of the operators involved and the assumptions made.

Vector Field $\mathbf{F}$

Let $\mathbf{F} : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field with components:

$$
\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))
$$

where each F_i is a real-valued function defined on Ω.

Assumptions on $\mathbf{F}$

- $\mathbf{F}$ is continuous on Ω,
- $\mathbf{F}$ has continuous second-order partial derivatives on Ω, i.e., $F_i \in C^2(\Omega)$.

These assumptions ensure the validity of Schwarz's theorem (also known as Clairaut's theorem) on the equality of mixed partial derivatives, which is central to the proof.

Differential Operators

- Gradient: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$.

- Divergence: For a vector field $\mathbf{F} = (F_1, F_2, F_3)$,

$$
\text{div} \, \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.
$$

- Curl: For $\mathbf{F}$,

$$
\text{curl} \, \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}

\right).
\]

The Goal and Strategy of the Proof

The primary goal is to establish that:

\[
\boxed{
\text{div}(\text{curl} \, \mathbf{F}) = 0
}
\]

for all points in \(\Omega\), given the smoothness assumptions. The key idea relies on explicitly calculating the divergence of the curl, then showing that the resulting expression simplifies to zero via the symmetry of mixed partial derivatives.

Strategy Overview

1. Express \(\text{curl} \, \mathbf{F}\) explicitly in component form.
2. Calculate the divergence of \(\text{curl} \, \mathbf{F}\) by taking the divergence operator on the curl vector.
3. Use the properties of partial derivatives, especially the equality of mixed derivatives under the smoothness assumptions, to simplify the expression.
4. Conclude that the divergence of the curl is identically zero within \(\Omega\).

Step-by-Step Detailed Proof

Step 1: Write the curl explicitly

Given the vector field \(\mathbf{F} = (F_1, F_2, F_3)\), the curl is:

\[
\text{curl} \, \mathbf{F} = \left(C_1, C_2, C_3 \right),
\]

where:

\[
\begin{cases}
C_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \\
C_2 = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \\
C_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.
\end{cases}
\]

Step 2: Compute the divergence of the curl

Applying divergence to $\text{curl} \, \mathbf{F}$:

$$
\text{div}(\text{curl} \, \mathbf{F}) = \frac{\partial C_1}{\partial x} + \frac{\partial C_2}{\partial y} + \frac{\partial C_3}{\partial z}.
$$

Substitute the expressions for C_1, C_2, C_3:

$$
\begin{aligned}
\text{div}(\text{curl} \, \mathbf{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}.
\end{aligned}
$$

Step 3: Group similar terms and exploit symmetry of mixed derivatives

Notice that terms can be grouped as:

$$
\left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_1}{\partial z \partial y}\right) + \left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_2}{\partial x \partial z}\right).
$$

Because $\mathbf{F}$ has continuous second derivatives ($F_i \in C^2(\Omega)$), the mixed partial derivatives are equal:

$$
\frac{\partial^2 F_i}{\partial x_j \partial x_k} = \frac{\partial^2 F_i}{\partial x_k \partial x_j}.
$$

Thus, each difference cancels out:

$$
\frac{\partial^2 F_i}{\partial x_j \partial x_k} - \frac{\partial^2 F_i}{\partial x_k \partial x_j} = 0,
$$

for all (i, j, k). Therefore,

$$

\text{div}(\text{curl} \, \mathbf{F}) = 0.
\]

Step 4: Conclusion

The above algebraic manipulations, rooted in the symmetry of mixed derivatives guaranteed by the smoothness conditions, establish that:

\[
\boxed{
\text{div}(\text{curl} \, \mathbf{F}) = 0,
}
\]

for all points in \(\Omega\).

Additional Insights and Context

Significance of the Result

This identity is fundamental in vector calculus because it reveals an intrinsic property of vector fields that are sufficiently smooth. It implies that the curl operator, when followed by divergence, yields zero—highlighting a form of compatibility condition for vector fields.

Implications in Physics

- Electromagnetism: The Maxwell equations include the statement that \(\nabla \cdot \mathbf{B} = 0\) (magnetic fields are divergence-free). The identity that divergence of curl is zero supports the mathematical consistency of such laws.
- Fluid Mechanics: For incompressible flows, the velocity field \(\mathbf{v}\) satisfies

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