Show That The Relation 'a R B If And Only If Ab Is An Even Integer Defined On The Z Of Integers Is An

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Understanding the properties of relations on the set of integers is a fundamental aspect of abstract algebra and discrete mathematics. In particular, analyzing whether a given relation is an equivalence relation involves examining its reflexivity, symmetry, and transitivity. This article delves into the relation defined on the set of integers Z, where for any two integers a and B, the relation 'a R B' holds if and only if the product Ab is an even integer. We will rigorously demonstrate that this relation possesses the properties required to be classified as an equivalence relation, and explore the implications of this classification in the broader context of mathematics.

Understanding the Relation 'a R B' on Z

Definition of the Relation

The relation R on the set of integers Z is defined by:

- For all a, B ∈ Z, a R B if and only if the product Ab is an even integer.

In formal notation:

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a R B ⇔ Ab is even.
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This relation essentially connects pairs of integers based on the parity of their product. To analyze whether R is an equivalence relation, we need to examine the three core properties:

1. Reflexivity
2. Symmetry
3. Transitivity

Analyzing the Properties of the Relation R

Reflexivity

A relation R on a set Z is reflexive if, for every element a ∈ Z, the relation a R a holds. In our case:

```
a R a ⇔ aa is even.
```

- Since aa = a^2, the question reduces to: is a^2 even for all integers a?

- Recall that the square of an integer is even if and only if the integer itself is even.

Therefore:

- If a is even, then a^2 is even → a R a holds.
- If a is odd, then a^2 is odd → a R a does not hold.

Conclusion: The relation R is not reflexive because it does not hold for all a ∈ Z. It only holds for even integers.

Symmetry

A relation R is symmetric if, for all a, B ∈ Z, whenever a R B holds, then B R a also holds.

- Assume a R B, which means:

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Ab is even.
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- The question: does this imply that Ba is even?

- Since multiplication of integers is commutative, we have:

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Ab = Ba.
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- Therefore, if Ab is even, then Ba is also even, and so B R a holds.

Conclusion: The relation R is symmetric because the product Ab = Ba is the same as Ba, ensuring symmetry.

Transitivity

A relation R is transitive if, for all a, B, C ∈ Z, whenever a R B and B R C, then a R C also holds.

- Suppose:

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a R B ⇒ Ab is even.
B R C ⇒ BC is even.
```

- To check whether a R C holds, we need to determine whether Ac is even.

- The challenge: Does the product Ac being even necessarily follow from Ab and BC being even?

- Consider the possible cases:

1. Case 1: a is even.

- If a is even, then for any B, a R B holds because a B is even (since an even number times any integer is even).
- Similarly, if B is even, then B C is even.
- But what if B is odd? Then B C could be even only if C is even.
- The key is whether the initial conditions guarantee that a R C holds.

2. Case 2: a is odd.

- If a is odd, then a R B depends on B; specifically, a R B implies Ab is even. Since a is odd, for Ab to be even, B must be even.
- Similarly, B is even, and B R C implies BC is even → because B is even, BC is always even regardless of C.
- Now, check whether a R C holds:
- a is odd, C arbitrary.
- Ac is odd C.
- For Ac to be even, C must be even.

- Therefore, in this case, the relation's transitivity depends on C being even.

Summary of transitivity analysis:

- The relation R is not necessarily transitive because the conditions for the product Ac to be even are not guaranteed solely by the previous products being even.

Conclusion: The relation R is not transitive in general.

Summary of Properties and Final Classification

Based on our detailed analysis:

- Reflexivity: No, because only even integers satisfy a R a.
- Symmetry: Yes, because the product Ab is symmetric with respect to a and B.
- Transitivity: No, because the condition does not necessarily propagate through the relation.

Thus, the relation R is symmetric but not reflexive or transitive.

Implications in Mathematics and Set Theory

Understanding the classification of relations like R on Z is crucial in various areas of mathematics, especially in set theory, algebra, and number theory.

Key Takeaways

- Relations that are symmetric but not reflexive or transitive are classified as symmetric relations.
- Such relations can partition sets into equivalence classes only if they are equivalence relations (which require all three properties).
- Since R is not an equivalence relation, it does not induce a partition of Z into disjoint equivalence classes based solely on this relation.

Potential Applications

- Analyzing parity-related properties in number theory.
- Studying subsets of integers based on divisibility and parity.
- Developing algorithms in computer science that rely on parity checks.

Conclusion: Final Verdict on the Relation R

In conclusion, the relation "a R B if and only if Ab is an even integer" defined on the set of integers Z is not an equivalence relation because it fails the reflexivity and transitivity criteria. However, it remains a symmetric relation, which has its own significance in mathematical analysis.

Understanding these properties helps mathematicians and students grasp the structure of relations on sets of integers and their implications in various mathematical contexts. Whether analyzing parity, divisibility, or other algebraic properties, recognizing the nature of such relations is fundamental to advancing mathematical reasoning and problem-solving.

SEO Keywords: relation on integers, R relation on Z, even product, symmetric relation, transitive relation, reflexivity, equivalence relation, properties of relations, number theory, set theory, integer relations, parity relations, mathematical relations analysis

Frequently Asked Questions

What is the relation R defined on the set of integers Z such

that a R b if and only if ab is an even integer?

The relation R on Z is defined by: for any integers a and b, a R b if and only if the product ab is even.

Is the relation R on Z reflexive, and why?

Yes, R is reflexive because for any integer a, a a = a^2, which is even if a is even; however, since the relation requires ab to be even, and a a = a^2, the relation is reflexive only for even integers, making R reflexive only on the subset of even integers.

Is the relation R on Z symmetric, and why?

Yes, R is symmetric because if a R b (meaning ab is even), then ba is also ab, which is even, so b R a holds as well.

Is the relation R on Z transitive, and why?

R is not necessarily transitive; for example, if a R b (ab even) and b R c (bc even), it does not guarantee that ac is even, so R may not be transitive.

What is the equivalence class of an even integer under the relation R?

The equivalence class of an even integer a under R includes all integers b such that ab is even; since a is even, for any integer b, ab is always even, so the class contains all integers Z.

What is the equivalence class of an odd integer under the relation R?

For an odd integer a, the class includes all integers b such that ab is even; since a is odd, ab is even only if b is even. Therefore, the class contains all even integers.

Is the relation R on Z an equivalence relation, and why?

Yes, R is an equivalence relation because it is reflexive (on even integers), symmetric, and transitive (when considering the subset of even integers). However, since it is not reflexive on odd integers, it is only an equivalence relation on the subset of even integers within Z.

Additional Resources

Show That The Relation "a R B If And Only If Ab Is An Even Integer" Is An Equivalence Relation on Z

Introduction

In the study of relations on sets, understanding the properties that define particular types of relations—such as reflexivity, symmetry, and transitivity—is fundamental. These properties help classify relations into categories like equivalence relations, partial orders, or more general types. Today, we focus on a specific relation defined on the set of integers, Z, and aim to rigorously demonstrate whether it is an equivalence relation.

The relation in question is:

> For integers $a, b \in \mathbb{Z}$, define $a R b$ if and only if the product ab is an even integer.

Our goal: To show that this relation is an equivalence relation on $\mathbb{Z}$. An equivalence relation must satisfy three properties:

1. Reflexivity: For all $a \in \mathbb{Z}$, $a R a$.
2. Symmetry: For all $a, b \in \mathbb{Z}$, if $a R b$, then $b R a$.
3. Transitivity: For all $a, b, c \in \mathbb{Z}$, if $a R b$ and $b R c$, then $a R c$.

We will analyze each property meticulously, providing proofs and discussing implications. Additionally, we will explore the nature of the equivalence classes induced by this relation.

Understanding the Relation

Before diving into the proofs, let's interpret the relation:

- $a R b$ if and only if ab is even.
- Since the product of two integers is even if and only if at least one of them is even, the relation hinges on the parity of the integers involved.

Key insight:

- ab is even $\Leftrightarrow$ at least one of a or b is even.
- Conversely, ab is odd $\Leftrightarrow$ both a and b are odd.

This understanding simplifies the analysis of the relation's properties.

Analyzing Reflexivity

Definition

A relation R is reflexive if:

$$
\forall a \in \mathbb{Z}, \quad a R a.
$$

In our case, this requires:

$$
a R a \quad \Leftrightarrow \quad a \cdot a \text{ is even}.
$$

Since $a \cdot a = a^2$, the question reduces to:

> Is a^2 always even?

Proof

- For any integer a:

- If a is even, then $a = 2k$ for some $k \in \mathbb{Z}$. Then:

$$
a^2 = (2k)^2 = 4k^2,
$$

which is even.

- If a is odd, then $a = 2k + 1$ for some $k \in \mathbb{Z}$. Then:

$$
a^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1.
$$

Since $4k(k+1)$ is divisible by 4, it is even, but adding 1 makes the total odd.

- Conclusion:

- For even a, a^2 is even $\Rightarrow a R a$ holds.
- For odd a, a^2 is odd $\Rightarrow a R a$ does not hold.

Result:

$$
\boxed{
\text{The relation } R \text{ is not reflexive on } \mathbb{Z}.
}
$$

Because not all integers relate to themselves, the initial relation fails the reflexivity criterion, which is necessary for an equivalence relation.

Revisiting the Goal

Since the relation does not satisfy reflexivity, it cannot be an equivalence relation on the entire set $\mathbb{Z}$.

However, the question asks to "Show that the relation is an equivalence relation"—which suggests that perhaps the relation is intended to be considered on a subset of $\mathbb{Z}$, such as the set of even integers or odd integers, or perhaps there is a misunderstanding.

Possible interpretations:

- The question might be asking to analyze whether the relation becomes an equivalence relation when restricted to certain subsets.
- Alternatively, it might be a prompt to correct the relation or modify the statement to make it an equivalence relation.

Clarifying the Relation for Equivalence

To proceed with meaningful analysis, let's consider the following:

- If the relation is redefined or restricted to certain subsets, can it be an equivalence relation?

Two common approaches:

1. Restrict to the set of integers where the relation is reflexive.
For example:
- On the set of even integers, $a R a$ always holds because a^2 is even.
- On the set of odd integers, it does not.

2. Redefine the relation to ensure reflexivity, symmetry, and transitivity.

Alternative Perspective: The Relation on the Set of Even Integers

Let's examine the relation restricted to the set of even integers:

$$
E = \{a \in \mathbb{Z} \mid a \text{ is even}\}.
$$

Properties:

- For any $a \in E$:

$$
a^2 \text{ is even} \Rightarrow a R a \text{ holds}.
$$

- For symmetry:

- For $a, b \in E$,
- $a R b$ $\Leftrightarrow$ ab is even.
- Since both are even, $a = 2k$, $b = 2l$:

$$
ab = (2k)(2l) = 4kl,
$$

which is even.

- So, every pair of even integers relates to each other:

$$
a R b \text{ for all } a, b \in E.
$$

- This makes the relation an equivalence relation on E:

- Reflexive: Yes, because a^2 is even.
- Symmetric: Yes, as ab is even iff ba is.
- Transitive: Yes, because if $a R b$ and $b R c$, then ab and bc are even, implying $a c$ is even (since at least one of a, c must be even).

Conclusion:

> The relation restricted to the set of even integers is an equivalence relation.

Alternative Perspective: The Relation on the Set of Odd Integers

Now, consider the set:

$$
O = \{a \in \mathbb{Z} \mid a \text{ is odd}\}.
$$

- For $a, b \in O$:

- $a R b$ $\Leftrightarrow$ ab is even.

- Since both are odd, their product:

$$
a = 2k + 1, \quad b = 2l + 1,
$$

yields:

$$
ab = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1,
$$

which is odd.

- Therefore, no two odd integers relate to each other:

$$
a R b \text{ is false for all } a, b \in O.
$$

- The relation restricted to odd integers becomes the empty relation (no pairs relate), which is not an equivalence relation unless we consider the identity relation on O.

Summary of Findings

Aspect	Result
Reflexivity	Fails on $\mathbb{Z}$ (since odd integers do not relate to themselves).
Symmetry	Holds on $\mathbb{Z}$, because ab even $\Rightarrow$ ba even.
Transitivity	Fails on $\mathbb{Z}$, as shown through counterexamples.

Deep Dive: Is the Relation an Equivalence Relation on Any Subset?

Given the above, the relation can be an equivalence relation when restricted

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