Show That Z Is A Standard Normal Random Variable; Then, For X > 0, (a) P{Z > X} = P{Z < -x};

Show That Z Is A Standard Normal Random Variable; Then, For X > 0, (a) P{Z > X} = P{Z < -x};

Understanding probability distributions is fundamental in statistics and probability theory. Among the most pivotal distributions is the standard normal distribution, often denoted as Z. Demonstrating that a particular random variable Z follows a standard normal distribution involves verifying its probability density function (PDF), cumulative distribution function (CDF), and symmetry properties. Moreover, exploring the symmetry inherent in the standard normal distribution reveals intriguing relationships such as the equality of tail probabilities for positive and negative values. This article aims to rigorously show that Z is a standard normal random variable and then, for X > 0, establish the relationship P{Z > X} = P{Z < -X}.

Understanding the Standard Normal Distribution

Before diving into proofs and properties, it is essential to comprehend what defines a standard normal distribution.

Definition and Characteristics

A standard normal random variable, Z, is a continuous random variable characterized by the following properties:

- Its probability density function (PDF) is given by:

$$
f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{ -\frac{z^2}{2} }, \quad -\infty < z < \infty
$$

- Its mean $\mu = 0$ and variance $\sigma^2 = 1$.

- The distribution is symmetric about the mean, i.e., $f_Z(z) = f_Z(-z)$.

- The cumulative distribution function (CDF), denoted as $\Phi(z)$, satisfies:

$$
\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} f_Z(t) \, dt
$$

Understanding these properties allows us to verify that a given Z follows this distribution.

Showing That Z Is a Standard Normal Random Variable

To demonstrate that Z is standard normal, one must typically show that its PDF matches the standard form and that it satisfies the properties outlined above.

Method 1: Derivation from a Known Distribution

If Z is derived from a process involving transformations of known distributions, such as the sum of independent normal variables, then:

- Since the sum of independent normal variables is normal, and scaling and shifting preserve normality, Z inherits the standard normal form if appropriately standardized.

Example:

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. normal variables each with mean μ and variance σ^2. The sample mean:

$$
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i
$$

has distribution $N(\mu, \sigma^2/n)$. Standardizing:

$$
Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}
$$

yields a standard normal distribution, i.e., $Z \sim N(0,1)$.

Conclusion: When Z is constructed as above, it follows the standard normal distribution.

Method 2: Direct Verification via PDF

Alternatively, if the probability density function of Z is explicitly given, verify that it matches the standard form:

$$
f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{ -\frac{z^2}{2} }
$$

This involves checking whether the PDF integrates to 1 and whether it exhibits symmetry.

Properties of the Standard Normal Distribution

An essential feature of the standard normal distribution is its symmetry about zero. This symmetry leads to significant probability relationships, especially concerning tail probabilities.

Symmetry Property

The PDF satisfies:

$$
f_Z(z) = f_Z(-z)
$$

for all z. Correspondingly, the CDF satisfies:

$$
\Phi(-z) = 1 - \Phi(z)
$$

which indicates that the probability of Z being less than $-z$ is equal to the probability of Z exceeding z.

Establishing the Relationship: For X > 0, P{Z > X} = P{Z < -X}

Using the properties of the standard normal distribution, particularly symmetry, we now demonstrate that:

$$
P(Z > X) = P(Z < -X), \quad \text{for } X > 0
$$

This relationship emphasizes the distribution's symmetry and is fundamental in statistical hypothesis testing and probability calculations.

Step-by-Step Proof

1. Express the probability $P(Z > X)$:

$$
P(Z > X) = 1 - P(Z \leq X) = 1 - \Phi(X)
$$

2. Express the probability $P(Z < -X)$:

$$
P(Z < -X) = \Phi(-X)
$$

3. Apply the symmetry property of the CDF:

$$
\Phi(-X) = 1 - \Phi(X)
$$

4. Conclude the equality:

$$
P(Z > X) = 1 - \Phi(X) = \Phi(-X) = P(Z < -X)
$$

Therefore:

$$
\boxed{
P(Z > X) = P(Z < -X), \quad \text{for } X > 0
}
$$

Implications and Applications of the Symmetry Property

Understanding that $P(Z > X) = P(Z < -X)$ for $X > 0$ has several practical implications:

- Tail probabilities: It simplifies calculations of tail probabilities in the standard normal distribution.

- Two-tailed tests: In hypothesis testing, especially when testing for deviations in either direction, this symmetry is crucial.

- Confidence intervals: Constructing symmetric confidence intervals around the mean relies on this property.

- Probability approximations: For large X, the probability of Z exceeding X can be directly related to the probability of Z being less than $-X$, simplifying calculations.

Conclusion

In this comprehensive overview, we have demonstrated how Z can be characterized as a standard normal random variable through its probability density function, symmetry, and derivation from known distributions. The critical property that $P(Z > X) = P(Z < -X)$ for $X > 0$ hinges on the symmetry of the standard normal distribution. Recognizing and leveraging this property simplifies many statistical calculations and underpins fundamental concepts in hypothesis testing, confidence interval construction, and probabilistic analysis. The symmetry inherent in the standard normal distribution not only facilitates mathematical convenience but also provides intuitive understanding of the behavior of normally distributed variables around their mean.

References:

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- Ross, S. M. (2014). Introduction to Probability and Statistics. Academic Press.

- Devore, J. L. (2011). Probability and Statistics for Engineering and the Sciences. Brooks/Cole.

Frequently Asked Questions

How can we show that Z is a standard normal random variable?

Z is a standard normal random variable if it has a mean of 0, a variance of 1, and its probability density function is symmetric about zero. Typically, Z is defined as $Z = (X - \mu) / \sigma$, where X is normally distributed with mean μ and standard deviation σ. When $\mu = 0$ and $\sigma = 1$, Z follows a standard normal distribution.

What is the symmetry property of the standard normal distribution?

The standard normal distribution is symmetric about zero, which means that for any real number x, $P(Z > x) = P(Z < -x)$. This symmetry is key to understanding the relationship between tail probabilities.

How do we interpret the equality P{Z > x} = P{Z < -x} for a standard normal variable?

This equality reflects the symmetry of the standard normal distribution about zero. It indicates that the probability of Z exceeding a positive value x is equal to the probability of Z being less than -x.

Why does the relation P{Z > x} = P{Z < -x} hold for Z ~ N(0,1)?

Because the standard normal distribution is symmetric around zero, the area under the curve to the right of x is equal to the area to the left of -x, making these probabilities equal.

For a standard normal variable Z, how is P(Z > X) related to P(Z < -X) when X > 0?

They are equal, i.e., $P(Z > X) = P(Z < -X)$, due to the symmetry of the standard normal distribution about zero.

Can you explain the significance of the relation P{Z > X} = P{Z < -X} in statistical inference?

Yes, this relation is used to simplify calculations involving tail probabilities, especially in hypothesis testing and confidence interval constructions, by leveraging the symmetry of the standard normal distribution.

How does understanding the symmetry property of Z help in calculating tail probabilities?

It allows us to relate probabilities on opposite tails of the distribution, simplifying calculations and enabling easier estimation of tail probabilities without complex integrals.

Additional Resources

Show That Z Is A Standard Normal Random Variable; Then, For X > 0, (a) P{Z > X} = P{Z < -X}

Introduction

In the vast landscape of probability theory and statistics, the normal distribution stands as one of the most fundamental and widely used probability models. Its elegant bell-shaped curve not only captures natural phenomena such as heights, test scores, and measurement errors but also forms the backbone of inferential statistics, including hypothesis testing and confidence intervals. Central to understanding the normal distribution is the concept of the standard normal random variable, often denoted as Z, which has a mean of zero and a variance of one.

This article aims to rigorously demonstrate that a given random variable Z is, in fact, a standard normal variable. Subsequently, it explores a pivotal symmetry property of the standard normal distribution: for any positive value X, the probability that Z exceeds X equals the probability that Z is less than -X. This symmetry underpins many statistical techniques and helps in understanding the behavior of tail probabilities, critical in areas such as risk assessment and quality control.

Defining the Standard Normal Random Variable

What Is a Standard Normal Variable?

A standard normal random variable Z is a continuous random variable characterized by its probability density function (pdf):

$$
f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad z \in (-\infty, \infty).
$$

This specific distribution is a special case of the more general normal distribution $N(\mu, \sigma^2)$, where:

- Mean (μ) = 0,
- Variance (σ^2) = 1.

The importance of Z being "standard" lies in its role as a reference distribution. Many normal variables can be transformed into Z through the standardization process, enabling easier calculations and comparisons.

Demonstrating That Z Is a Standard Normal Variable

Step 1: Starting Point – The General Normal Distribution

Suppose $X \sim N(\mu, \sigma^2)$. Its probability density function is:

$$
f_X(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x - \mu)^2}{2\sigma^2}}.
$$

Standardizing this variable involves transforming it into a new variable Z:

$$
Z = \frac{X - \mu}{\sigma}.
$$

This transformation shifts the mean to zero and scales the variance to one.

Step 2: Distribution of Z

To find the distribution of Z, we derive its pdf:

$$
f_Z(z) = P(Z \leq z) \implies \text{by change of variables:}
$$
$$
f_Z(z) = f_X(\mu + \sigma z) \times \left| \frac{d}{dz} (\mu + \sigma z) \right| = f_X(\mu + \sigma z) \times \sigma.
$$

Substituting the pdf of X:

$$
f_Z(z) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(\mu + \sigma z - \mu)^2}{2\sigma^2}} \times \sigma = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.
$$

This is precisely the pdf of the standard normal distribution, confirming that:

$$
Z = \frac{X - \mu}{\sigma} \sim N(0, 1).
$$

\]

Thus, standardization transforms any normal variable into a standard normal variable.

Symmetry Property of the Standard Normal Distribution

Having established that Z is a standard normal variable, we now explore a key property: the symmetry of its distribution.

The Symmetry Intuition

The standard normal distribution is symmetric about zero. The density function satisfies:

$$
f_Z(-z) = f_Z(z),
$$

for all z. This symmetry implies that the distribution of Z "mirrors" itself across the vertical axis at zero.

Formal Proof: $P\{Z > x\} = P\{Z < -x\}$ for $x > 0$

Let's consider the probabilities:

$$
P\{Z > x\} \quad \text{and} \quad P\{Z < -x\}.
$$

Because the distribution is symmetric:

$$
P\{Z > x\} = \int_x^\infty f_Z(z) \, dz,
$$

$$
P\{Z < -x\} = \int_{-\infty}^{-x} f_Z(z) \, dz.
$$

By substitution, note that:

$$
\int_{-\infty}^{-x} f_Z(z) \, dz = \int_{x}^{\infty} f_Z(-z) \, dz,
$$

since changing variable $z' = -z$, $dz' = -dz$, and the limits switch accordingly.

Using the symmetry $f_Z(-z) = f_Z(z)$:

$$
P\{Z < -x\} = \int_{x}^{\infty} f_Z(z) \, dz = P\{Z > x\}.
$$

Therefore, for any $x > 0$:

$$
\boxed{
P\{Z > x\} = P\{Z < -x\}.
}
$$

This equality reflects the intuitive fact that the probability of Z falling into the upper tail beyond x is equal to the probability of it falling into the lower tail below -x.

Practical Implications and Applications

The symmetry property of the standard normal distribution is more than a mathematical curiosity; it has vital practical applications across various fields.

1. Simplification of Tail Probability Calculations

When calculating tail probabilities, especially for large values of x, the symmetry allows statisticians to focus on either the upper or lower tail, knowing the other mirrors it. This reduces computational complexity and makes probabilistic bounds easier to derive.

2. Confidence Intervals and Significance Testing

In hypothesis testing, critical values often correspond to tail probabilities. The symmetry ensures that the critical value $z_{ \alpha/2 }$ satisfies:

$$
P\{Z > z_{ \alpha/2 }\} = \frac{\alpha}{2} = P\{Z < -z_{ \alpha/2 }\},
$$

which underpins two-tailed tests, commonly used to assess deviations from a null hypothesis.

3. Risk Management and Extreme Events

In fields like finance, engineering, and environmental science, understanding the likelihood of extreme events involves tail probability estimation. The symmetry property simplifies assessing risks of deviations in either direction.

Extending Beyond the Standard Normal

While the article focuses on the standard normal distribution, the concepts extend seamlessly to any normal variable through the standardization process:

$$
X \sim N(\mu, \sigma^2) \implies Z = \frac{X - \mu}{\sigma} \sim N(0, 1).
$$

The symmetry property holds for Z, and by inverse transformation, similar properties can be inferred for X.

Conclusion

In summary, the demonstration that Z is a standard normal variable hinges on the process of standardization, transforming any normal distribution into a canonical form with mean zero and variance one. This makes Z a universal reference for probabilistic calculations involving normal variables.

Furthermore, the distribution's inherent symmetry about zero manifests in the elegant equality:

$$
P\{Z > x\} = P\{Z < -x\},
$$

for all $x > 0$. This property is not only theoretically pleasing but also practically invaluable, simplifying calculations, guiding statistical inference, and underpinning the understanding of tail behaviors in diverse applications.

The standard normal distribution's symmetry thus remains a cornerstone in the edifice of probability theory, illuminating the balanced nature of many natural and social phenomena, and providing a powerful tool for statisticians and analysts worldwide.

Show That Z Is A Standard Normal Random Variable Then For X Gt 0 A P Z Gt X P Z Lt X

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