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Absolute value equations are fundamental in algebra, representing distances on the number line regardless of direction. These equations are essential because they help in solving real-world problems involving magnitudes, such as measuring deviations, calculating tolerances, or understanding symmetry. When solving absolute value equations, the key idea is to consider both the positive and negative scenarios that satisfy the equation, since the absolute value of a number is always non-negative. This article provides a comprehensive guide to solving various types of absolute value equations, illustrating methods and common pitfalls to avoid. Whether you're a student preparing for exams or someone seeking to deepen your understanding, this in-depth exploration aims to clarify the process step-by-step.

Understanding Absolute Value Equations

What Is an Absolute Value?

The absolute value of a number is its distance from zero on the number line, regardless of direction. It is denoted by vertical bars, $|x|$. For example:

- $|5| = 5$
- $|-5| = 5$

By definition, the absolute value of a real number x is:

$|x| = x$, if $x \geq 0$
$|x| = -x$, if $x < 0$

Types of Absolute Value Equations

Absolute value equations generally take forms such as:

1. $|ax + b| = c$
2. $|ax + b| < c$
3. $|ax + b| > c$
4. $|ax + b| \le c$

In this article, our focus is primarily on equations of the form:

$|ax + b| = c$

where a, b, and c are real numbers and $c \ge 0$. The goal is to find all real solutions for x.

Solving Basic Absolute Value Equations

Step-by-Step Approach

To solve $|ax + b| = c$, follow these steps:

1. Isolate the absolute value expression if necessary.
2. Set up two separate equations to account for the positive and negative cases:
   - $ax + b = c$
   - $ax + b = -c$
3. Solve each equation for x.
4. Check for extraneous solutions, especially if the original equation involves restrictions or extraneous conditions.

Example 1: Solving $|2x + 3| = 7$

1. Write two equations:
    - $2x + 3 = 7$
    - $2x + 3 = -7$

2. Solve each:
    - $2x + 3 = 7 \rightarrow 2x = 4 \rightarrow x = 2$
    - $2x + 3 = -7 \rightarrow 2x = -10 \rightarrow x = -5$

3. Solutions: $x = 2$, $x = -5$

Handling Absolute Values with Inequalities

Solving $|ax + b| < c$ (Less Than)

This inequality means the expression's absolute value is less than a certain number, indicating the expression lies within a range around zero.

Solution approach:

1. Rewrite as a compound inequality:

    $-c < ax + b < c$

2. Solve for x:
    - $ax + b > -c \rightarrow ax > -c - b \rightarrow x > (-c - b)/a$ (if $a > 0$)
    - $ax + b < c \rightarrow ax < c - b \rightarrow x < (c - b)/a$ (if $a > 0$)

3. If $a < 0$, reverse the inequalities when dividing by a.

4. Solution set is the intersection of the two inequalities.

Example 2: Solving $|3x - 4| < 5$

1. Rewrite:

   $-5 < 3x - 4 < 5$

2. Break into two inequalities:

   - $3x - 4 > -5 \rightarrow 3x > -1 \rightarrow x > -1/3$
   - $3x - 4 < 5 \rightarrow 3x < 9 \rightarrow x < 3$

3. Solution: $-1/3 < x < 3$

Solving $|ax + b| > c$ (Greater Than)

This indicates the expression's absolute value exceeds a certain value, leading to two separate solution regions:

1. $ax + b > c$
2. $ax + b < -c$

Note: When solving inequalities involving absolute values, carefully watch for the direction of inequalities when dividing by negative numbers.

Special Cases and Additional Considerations

Case 1: When $c = 0$

The equation $|ax + b| = 0$ simplifies to $ax + b = 0$, since the absolute value of a number is zero only when the number itself is zero.

- Solution: $x = -b/a$ (assuming $a \neq 0$)
- If $a = 0$ and $b \neq 0$, then $|b| \neq 0$, so no solution.

- If both a and b are zero, the equation becomes $0 = 0$, which is true for all real x, leading to infinitely many solutions.

Case 2: When the expression inside the absolute value is zero

If the expression inside the absolute value is zero, then the solution is straightforward: set $ax + b = 0$ and solve for x.

Handling No Solution Scenarios

For equations like $|ax + b| = -c$ where $c > 0$, there are no solutions because absolute value cannot be negative. Recognizing these scenarios early prevents unnecessary calculations.

Summary of Solution Strategies

- Always check the value of c before proceeding:
  - If $c < 0$, no solutions.
  - If $c = 0$, solution is simply $x = -b/a$ (if $a \neq 0$).
  - If $c > 0$, proceed with the standard method.
- When solving $|ax + b| = c$, set up two equations and solve each.
- When solving inequalities involving absolute values, convert to double inequalities and solve each part carefully, considering the sign of a.
- Always verify solutions, especially when dividing by negative numbers.

Practice Problems and Their Solutions

Problem 1: Solve $|4x - 5| = 9$

Solution:

1. Set up two equations:
    - $4x - 5 = 9 \rightarrow 4x = 14 \rightarrow x = 14/4 = 7/2$
    - $4x - 5 = -9 \rightarrow 4x = -4 \rightarrow x = -1$
2. Solutions: $x = 7/2, -1$

Problem 2: Solve $|x + 2| < 3$

Solution:

1. Rewrite:

   $-3 < x + 2 < 3$
2. Solve each:
    - $x + 2 > -3 \rightarrow x > -5$
    - $x + 2 < 3 \rightarrow x < 1$
3. Solution: $-5 < x < 1$

Problem 3: Solve $|5x - 1| > 7$

Solution: