Solve The Following Differential Equation Problem For X(t) Using Laplace Transforms X + 2x = E-t X(0)

Solve The Following Differential Equation Problem For X(t) Using Laplace Transforms X + 2x = E-t X(0)

Introduction

Differential equations are fundamental tools in mathematics and engineering, modeling a wide array of phenomena from physics to economics. Among various methods to solve these equations, Laplace transforms stand out as a powerful technique, especially for equations involving initial conditions and non-homogeneous terms.

In this article, we will explore the process of solving the differential equation:

$$
X' + 2X = e^{-t} X(0)
$$

using Laplace transforms. We will delve into the theoretical background, step-by-step solution procedures, and practical insights to help you understand and master this method. Whether you are a student preparing for exams or a professional solving real-world problems, this comprehensive guide aims to provide clarity and depth.

Understanding the Differential Equation

The Equation in Context

The given differential equation is:

$$
X' + 2X = e^{-t} X(0)
$$

with an initial condition $X(0)$. Here, $X(t)$ is a function of time t, and X' denotes its derivative with respect to t.

This is a first-order linear differential equation with a non-homogeneous term involving an exponential function multiplied by the initial value $X(0)$. The presence of $X(0)$ on the right side makes this an initial value problem, which can be effectively handled using Laplace transforms.

Why Use Laplace Transforms?

Laplace transforms convert differential equations from the time domain into algebraic equations in the complex frequency domain (s-domain). This simplifies the process of solving differential equations by turning derivatives into algebraic terms.

Key advantages include:
- Handling initial conditions naturally.
- Simplifying the solution process for equations with exponential, sinusoidal, or step functions.
- Facilitating solutions to equations with discontinuities or impulses.

Fundamentals of Laplace Transforms

Definition

The Laplace transform of a function \(f(t)\), defined for \(t \geq 0\), is:

\[
\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt
\]

where \(s\) is a complex number.

Common Laplace Transform Pairs

- \(\mathcal{L}\{1\} = \frac{1}{s}\)
- \(\mathcal{L}\{e^{at}\} = \frac{1}{s - a}\)
- \(\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}\)
- \(\mathcal{L}\{\delta(t - a)\} = e^{-as}\)

Derivative Property

The Laplace transform of a derivative:

\[
\mathcal{L}\{X'(t)\} = sX(s) - X(0)
\]

This property allows us to incorporate initial conditions directly into the transformed equation.

Step-by-Step Solution Using Laplace Transforms

Step 1: Take the Laplace Transform of Both Sides

Applying the Laplace transform to the differential equation:

\[

$$
X' + 2X = e^{-t}X(0)
$$

we get:

$$
\mathcal{L}\{X'\} + 2 \mathcal{L}\{X\} = \mathcal{L}\{e^{-t} X(0)\}
$$

Using the derivative property:

$$
sX(s) - X(0) + 2X(s) = X(0) \mathcal{L}\{e^{-t}\}
$$

Step 2: Simplify the Transformed Equation

Recall:

$$
\mathcal{L}\{e^{-t}\} = \frac{1}{s + 1}
$$

Substituting:

$$
sX(s) - X(0) + 2X(s) = X(0) \cdot \frac{1}{s + 1}
$$

Rearranged:

$$
(s + 2) X(s) - X(0) = \frac{X(0)}{s + 1}
$$

Step 3: Solve for $X(s)$

Isolate $X(s)$:

$$
(s + 2) X(s) = X(0) + \frac{X(0)}{s + 1}
$$

$$
X(s) = \frac{X(0)}{s + 2} + \frac{X(0)}{(s + 2)(s + 1)}
$$

This expression gives the Laplace transform of the solution in terms of s and initial condition $X(0)$.

Step 4: Perform Partial Fraction Decomposition

To find the inverse Laplace transform, decompose:

$$
X(s) = \frac{X(0)}{s + 2} + \frac{X(0)}{(s + 2)(s + 1)}
$$

Focus on the second term:

$$
\frac{X(0)}{(s + 2)(s + 1)} = \frac{A}{s + 2} + \frac{B}{s + 1}
$$

Multiply through to solve for A and B:

$$
X(0) = A(s + 1) + B(s + 2)
$$

Set $s = -1$:

$$
X(0) = A(0) + B(1) \Rightarrow B = X(0)
$$

Set $s = -2$:

$$
X(0) = A(-1) + B(0) \Rightarrow A = -X(0)
$$

Thus,

$$
\frac{X(0)}{(s + 2)(s + 1)} = -\frac{X(0)}{s + 2} + \frac{X(0)}{s + 1}
$$

Step 5: Write $X(s)$ in Simplified Form

Now,

$$
X(s) = \frac{X(0)}{s + 2} - \frac{X(0)}{s + 2} + \frac{X(0)}{s + 1}
$$

which simplifies to:

$$
X(s) = \frac{X(0)}{s + 1}
$$

Step 6: Find the Inverse Laplace Transform

The inverse Laplace transform:

$$
X(t) = X(0) e^{-t}
$$

Final Solution:

$$
\boxed{
X(t) = X(0) e^{-t}
}
$$

This indicates that the solution decays exponentially at a rate determined by the initial value $X(0)$.

Additional Insights and Interpretation

Verifying the Solution

Substitute $X(t) = X(0) e^{-t}$ into the original differential equation:

$$
X' + 2X = e^{-t} X(0)
$$

Calculate X':

$$
X' = -X(0) e^{-t}
$$

Plug into the equation:

$$
- X(0) e^{-t} + 2 X(0) e^{-t} = e^{-t} X(0)
$$

Simplify:

$$
X(0) e^{-t} (-1 + 2) = e^{-t} X(0)
$$

$$
X(0) e^{-t} = e^{-t} X(0)
$$

which confirms the solution is valid.

Significance of the Solution

- The solution demonstrates exponential decay, a common characteristic in systems like RC circuits, population dynamics, or radioactive decay.
- The initial condition $X(0)$ scales the entire solution, reflecting the system's starting state.

Practical Applications of Laplace Transform Solutions

Control Systems

Designing controllers for dynamic systems often involves solving differential equations where initial conditions are critical. Laplace transforms simplify the analysis and design process.

Electrical Engineering

Analyzing circuits with capacitors and inductors involves solving differential equations with exponential sources, making Laplace transforms invaluable.

Mechanical Vibrations

Modeling damping in mechanical systems involves similar equations where exponential decay solutions are expected.

Summary and Key Takeaways

- Laplace transforms convert differential equations into algebraic equations that are easier to solve.
- The method naturally incorporates initial conditions.
- Partial fraction decomposition is essential for inverse transforms.
- The solution process involves transforming, algebraically solving, and then applying inverse transforms.

Conclusion

Mastering the use of Laplace transforms for solving differential equations equips you with a robust tool for tackling complex dynamic systems. The process outlined in this article—transforming, algebraic manipulation, partial fractions, and inverse transformation—provides a systematic approach to solving first-order linear differential equations like $X' + 2X = e^{-t} X(0)$.

Understanding the solution's behavior, especially the exponential decay characteristic, deepens your insight into system dynamics. Whether in academic studies or practical engineering problems, proficiency with Laplace transforms enhances your analytical capabilities and problem-solving efficiency.

References

- O'Neil, P. V. (2011). Advanced Engineering Mathematics. Cengage Learning.
- Spiegel,

Frequently Asked Questions

How do you apply Laplace transforms to solve the differential equation X + 2x = e^{-t} with initial condition X(0)?

First, take the Laplace transform of both sides, using L{X(t)} = X(s) and L{dx/dt} = sX(s) - X(0). Convert the differential equation into algebraic form in the s-domain: (s + 2)X(s) - X(0) = 1/(s + 1). Then, solve for X(s) and take the inverse Laplace transform to find X(t).

What initial conditions are needed to fully solve the differential equation using Laplace transforms?

You need the initial value X(0) of the function X(t). Without this, the solution will be incomplete as it influences the transformed equation and the inverse transform.

How do you handle the term e^{-t} when applying Laplace transforms to the differential equation?

The Laplace transform of e^{-t} is 1/(s + 1). When this term appears on the right side of the transformed equation, it allows you to solve for X(s) directly by considering the algebraic relation involving this transform.

What is the general solution for X(t) after applying Laplace transforms to the given differential equation?

After solving for X(s) and applying the inverse Laplace transform, the solution typically involves exponential functions. For this specific problem, the solution is X(t) = C e^{-2t} + (1/3) e^{-t}, where C is determined by the initial condition X(0).

How do initial conditions influence the final solution in the Laplace transform method for this differential

equation?

Initial conditions determine the constant terms in the solution. Specifically, knowing X(0) allows you to compute the constant C in the inverse transform, ensuring the solution accurately reflects the initial state of the system.

Additional Resources

Differential Equations and Laplace Transforms: A Comprehensive Solution to X(t) = E - t X(0)

Introduction: Unlocking the Power of Laplace Transforms in Differential Equations

Differential equations are fundamental tools in engineering, physics, and applied mathematics, serving as the language through which we describe dynamic systems. Among the various methods to solve these equations, the Laplace Transform stands out for its efficiency, especially when dealing with initial value problems involving linear differential equations with constant coefficients.

In this article, we explore a specific example: solving the differential equation

$$ X'' + 2X' = e^{-t} \quad \text{with initial condition} \quad X(0) = 0 $$

using Laplace transforms. While the original problem statement appears somewhat ambiguous, it resembles a typical second-order linear differential equation with constant coefficients, subject to initial conditions. Our goal is to walk through each step meticulously, providing a comprehensive understanding of how Laplace transforms facilitate the solution process.

Understanding the Differential Equation

Equation Breakdown

The differential equation given is:

$$ X''(t) + 2 X'(t) = e^{-t} $$

with the initial condition:

$$ X(0) = 0 $$

Note: For clarity, we assume $X(t)$ is twice differentiable, and we're solving for $X(t)$.

This is a linear, nonhomogeneous second-order differential equation with constant coefficients, which is a common class of equations encountered in systems modeling, such as mechanical vibrations or electrical circuits.

Why Use Laplace Transforms?

The Laplace Transform method simplifies the process by converting differential equations in the time domain into algebraic equations in the complex frequency domain. This transformation turns derivatives into polynomial multipliers, making the algebraic manipulation more straightforward, especially when initial conditions are known.

Step-by-Step Solution Using Laplace Transforms

1. Applying the Laplace Transform

Recall the Laplace Transform definition:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st}f(t)dt
$$

And the key properties for derivatives:

$$
\mathcal{L}\{f'(t)\} = sF(s) - f(0)
$$
$$
\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)
$$

Assuming initial conditions $X(0) = 0$ and $X'(0) = 0$ for simplicity (if not given, we can specify these as zero; otherwise, they should be provided), the transformation becomes more straightforward.

Applying the Laplace Transform to both sides:

$$
\mathcal{L}\{X''(t) + 2X'(t)\} = \mathcal{L}\{e^{-t}\}
$$

which yields:

$$
s^2 X(s) - s X(0) - X'(0) + 2 (s X(s) - X(0)) = \frac{1}{s + 1}
$$

Since $X(0) = 0$ and $X'(0) = 0$:

$$
s^2 X(s) + 2s X(s) = \frac{1}{s + 1}
$$

---

2. Solving for $X(s)$

Combine like terms:

$$
(s^2 + 2s) X(s) = \frac{1}{s + 1}
$$

Express $X(s)$:

$$
X(s) = \frac{1}{(s + 1)(s^2 + 2s)}
$$

Factor the quadratic:

$$
s^2 + 2s = s(s + 2)
$$

So,

$$
X(s) = \frac{1}{(s + 1) \cdot s \cdot (s + 2)}
$$

---

3. Partial Fraction Decomposition

To invert $X(s)$, we need to decompose into simpler fractions:

\[

X(s) = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2}
\]

Multiply both sides by the denominator:

\[
1 = A(s + 1)(s + 2) + B s (s + 2) + C s (s + 1)
\]

Expand each term:

\[
A(s^2 + 3s + 2) + B(s^2 + 2s) + C(s^2 + s)
\]

Combine:

\[
1 = A s^2 + 3A s + 2A + B s^2 + 2 B s + C s^2 + C s
\]

Group like terms:

\[
1 = (A + B + C) s^2 + (3A + 2B + C) s + 2A
\]

Set coefficients equal to match the constant 1:

\[
\begin{cases}
A + B + C = 0 \\
3A + 2B + C = 0 \\
2A = 1
\end{cases}
\]

Solve the system:

From the third equation:

\[
A = \frac{1}{2}
\]

Substitute \(A = \frac{1}{2}\) into the first:

\[
\frac{1}{2} + B + C = 0 \Rightarrow B + C = -\frac{1}{2}
\]

Substitute into the second:

$$
3 \times \frac{1}{2} + 2B + C = 0 \Rightarrow \frac{3}{2} + 2B + C = 0
$$

Express C:

$$
C = -\frac{3}{2} - 2B
$$

Now, substitute into $B + C = -\frac{1}{2}$:

$$
B + \left(-\frac{3}{2} - 2B \right) = -\frac{1}{2}
$$

Simplify:

$$
B - \frac{3}{2} - 2B = -\frac{1}{2}
$$

$$
- B - \frac{3}{2} = -\frac{1}{2}
$$

$$
- B = -\frac{1}{2} + \frac{3}{2} = 1
$$

$$
B = -1
$$

Now, find C:

$$
C = -\frac{3}{2} - 2(-1) = -\frac{3}{2} + 2 = \frac{1}{2}
$$

Summary of partial fractions:

$$
A = \frac{1}{2}, \quad B = -1, \quad C = \frac{1}{2}
$$

Thus,

$$
X(s) = \frac{1/2}{s} - \frac{1}{s + 1} + \frac{1/2}{s + 2}
$$

---

4. Inverse Laplace Transform to Find $X(t)$

Using standard Laplace tables:

$$
\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1
$$
$$
\mathcal{L}^{-1}\left\{\frac{1}{s + a}\right\} = e^{-a t}
$$

Applying this to each term:

$$
X(t) = \frac{1}{2} \cdot 1 - 1 \cdot e^{-t} + \frac{1}{2} e^{-2t}
$$

or simplified:

$$
X(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}
$$

Final Solution and Interpretation

The explicit solution for $X(t)$ in terms of elementary functions is:

$$
\boxed{
X(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}
}
$$

This function describes the behavior of the system governed by the original differential equation, with initial condition $X(0) = 0$.

Additional Considerations

- Initial Conditions: If other initial conditions such as $X'(0)$ were specified, they would affect the Laplace transform equations and thus the solution.
- Homogeneous Solution: The complementary (homogeneous) solution is $X_h(t) = C_1 e^{0 \cdot t} + C_2 e^{-2t}$, but since initial conditions are specified, constants can be determined accordingly.
- Particular Solution: The nonhomogeneous term e^{-t} influences the particular solution, which we've obtained via algebraic manipulation.

Conclusion: The Elegance of Laplace Transforms in Differential Equation Solutions

This detailed walkthrough demonstrates the power and elegance of the Laplace Transform method

Solve The Following Differential Equation Problem For X T Using Laplace Transforms X 2x E T X 0

Solve The Following Differential Equation Problem For X T Using Laplace Transforms X 2x E T X 0

Related Articles

- The Objective Of A Firm In A Responsive Supply Chain Is Likely To Be Realized If It Has A Select One:
- Three Grams Of Musk Oil Are Required For Each Bottle Of Mink Caress, A Very Popular Perfume Made By A
- Suppose The Market Portfolio Is Equally Likely To Increase By 20% Or Decrease By 4%. A. Calculate The

Back to Home