Solve The Homogeneous Second-order Initial Value Problem: A) Y" + 2y' + 5y = 0,y (0) = 3,y'(0) = 5 B)

Solve The Homogeneous Second-order Initial Value Problem: A) Y" + 2y' + 5y = 0,y (0) = 3,y'(0) = 5 B)

Understanding how to solve second-order homogeneous differential equations with initial conditions is fundamental in both pure and applied mathematics, especially in physics and engineering. This article provides a comprehensive guide to solving the specific initial value problem (IVP):

Part A:
$$ y'' + 2y' + 5y = 0 $$
with initial conditions:
$$ y(0) = 3, \quad y'(0) = 5 $$

Additionally, we will explore the steps involved in solving such differential equations, the general solution, particular solutions using initial conditions, and key concepts essential for mastering homogeneous second-order IVPs.

Understanding Homogeneous Second-Order Differential Equations

What Is a Homogeneous Second-Order Differential Equation?

A second-order differential equation involves the second derivative of an unknown function $y(t)$, typically written as:

$$ y'' + p(t) y' + q(t) y = 0 $$

If the right side of the equation is zero, the equation is called homogeneous. For constant coefficients, the general form simplifies to:

$$ y'' + a y' + b y = 0 $$

where a and b are constants.

Significance of Homogeneity

Homogeneous equations are significant because their solutions involve exponential functions, which are straightforward to analyze and interpret physically. They often model systems with no external forces, such as free vibrations in mechanical systems.

Step-by-Step Solution to the Homogeneous Second-Order IVP

Step 1: Write the Differential Equation

Given:

$$ y'' + 2 y' + 5 y = 0 $$

This is a linear homogeneous differential equation with constant coefficients.

Step 2: Find the Characteristic Equation

To solve, assume a solution of the form:

$$ y(t) = e^{rt} $$

Substituting into the differential equation yields:

$$ r^2 e^{rt} + 2 r e^{rt} + 5 e^{rt} = 0 $$

Dividing through by e^{rt} (which is never zero):

$$ r^2 + 2 r + 5 = 0 $$

This quadratic equation is known as the characteristic equation.

Step 3: Solve the Characteristic Equation

Using the quadratic formula:

$$ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} $$

where $a=1, b=2, c=5$:

$$ r = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2} = \frac{-2 \pm \sqrt{4 - 20}}{2} $$

$$ r = \frac{-2 \pm \sqrt{-16}}{2} $$

$$ r = \frac{-2 \pm 4i}{2} $$

$$ r = -1 \pm 2i $$

The roots are complex conjugates, indicating oscillatory behavior.

Step 4: Write the General Solution

For roots $r = \alpha \pm \beta i$, the general solution is:

$$ y(t) = e^{\alpha t} (C_1 \cos \beta t + C_2 \sin \beta t) $$

Substituting $\alpha = -1$, $\beta=2$:

$$ y(t) = e^{-t} (C_1 \cos 2 t + C_2 \sin 2 t) $$

Applying Initial Conditions to Find Particular Solution

Step 5: Derive the First Derivative $y'(t)$

Differentiate $y(t)$:

$$
\begin{aligned}
y(t) &= e^{-t} (C_1 \cos 2 t + C_2 \sin 2 t) \\
y'(t) &= \frac{d}{dt} \left[e^{-t} (C_1 \cos 2 t + C_2 \sin 2 t) \right]
\end{aligned}
$$

Applying the product rule:

$$
\begin{aligned}
y'(t) &= e^{-t} \cdot \frac{d}{dt} (C_1 \cos 2 t + C_2 \sin 2 t) + \frac{d}{dt} (e^{-t}) \cdot (C_1 \cos 2 t + C_2 \sin 2 t) \\
&= e^{-t} \left(-2 C_1 \sin 2 t + 2 C_2 \cos 2 t \right) - e^{-t} (C_1 \cos 2 t + C_2 \sin 2 t)
\end{aligned}
$$

\]

Simplify:

\[
\boxed{
y'(t) = e^{-t} \left[-2 C_1 \sin 2 t + 2 C_2 \cos 2 t - C_1 \cos 2 t - C_2 \sin 2 t \right]
}
\]

Step 6: Apply Initial Conditions at \(t=0 \)

Given:

\[
\begin{cases}
y(0) = 3 \\
y'(0) = 5
\end{cases}
\]

Calculate \(y(0) \):

\[
y(0) = e^{0} (C_1 \cos 0 + C_2 \sin 0) = 1 \times (C_1 \times 1 + C_2 \times 0) = C_1
\]

Thus:

\[
C_1 = 3
\]

Calculate \(y'(0) \):

\[
\begin{aligned}
y'(0) &= e^{0} \left[-2 C_1 \sin 0 + 2 C_2 \cos 0 - C_1 \cos 0 - C_2 \sin 0 \right] \\
&= 1 \times \left[0 + 2 C_2 \times 1 - C_1 \times 1 - 0 \right] \\
&= 2 C_2 - C_1
\end{aligned}
\]

Substitute \(C_1=3 \) and set equal to 5:

\[
2 C_2 - 3 = 5

\]

Solve for C_2:

$$
2 C_2 = 8 \Rightarrow C_2 = 4
$$

Final Solution to the IVP

Putting everything together, the particular solution is:

$$
\boxed{
y(t) = e^{-t} \left(3 \cos 2 t + 4 \sin 2 t \right)
}
$$

This solution satisfies both the differential equation and the initial conditions.

Summary and Key Takeaways

- Homogeneous second-order differential equations with constant coefficients can be solved using the characteristic equation.
- Complex roots lead to oscillatory solutions involving exponential decay or growth modulated by sine and cosine.
- Initial conditions are used to determine the specific constants in the general solution, providing the particular solution relevant to the initial state.
- The process involves deriving the general solution, differentiating, and applying initial conditions systematically.

Additional Notes and Tips for Solving Similar Problems

- Always start by writing the characteristic equation and solving for roots.
- Recognize the form of roots: real and distinct, real and repeated, or complex conjugates.
- For complex roots $\alpha \pm \beta i$, remember the solution involves exponential decay/growth multiplied by sine and cosine functions.

- Use initial conditions to find the specific constants, ensuring the solution fits the initial state.
- Practice with various coefficients to become comfortable with different types of roots and their solutions.

Conclusion

Solving homogeneous second-order initial value problems is a fundamental skill in differential equations, enabling the modeling of diverse physical systems such as mechanical vibrations, electrical circuits, and more. The example discussed—solving $y'' + 2 y' + 5 y = 0$ with initial conditions—demonstrates the systematic approach: formulating the characteristic equation, solving for roots, writing the general solution, and applying initial conditions to find the particular solution. Mastery of these steps provides a solid foundation for tackling more complex differential equations and understanding the dynamic behavior of systems modeled by such equations.

Keywords: differential equations, homogeneous second-order differential equation, initial value problem, characteristic equation, complex roots, oscillatory solutions, exponential functions, physics modeling, engineering mathematics

Frequently Asked Questions

How do you solve the homogeneous second-order differential equation y'' + 2y' + 5y = 0 with initial conditions y(0) = 3 and y'(0) = 5?

To solve the differential equation, first find the characteristic equation: $r^2 + 2r + 5 = 0$. The roots are $r = -1 \pm 2i$. The general solution is $y(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$. Using initial conditions, substitute t=0: $y(0)= C_1 = 3$, and y'(0) to find C_2, resulting in the complete particular solution.

What is the general solution form for the differential equation y'' + 2y' + 5y = 0?

The general solution is $y(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$, where C_1 and C_2 are constants determined by initial conditions.

How do the initial conditions y(0) = 3 and y'(0) = 5 help

determine the specific solution to y'' + 2y' + 5y = 0?

By substituting t=0 into the general solution and its derivative, we get equations: $y(0) = C_1 = 3$ and $y'(0) = -C_1 + 2C_2 = 5$. Solving these yields $C_1=3$ and $C_2=4$, giving the particular solution y(t) = e^{-t}(3 cos 2t + 4 sin 2t).

What is the step-by-step process to find the solution to the initial value problem y'' + 2y' + 5y = 0 with given initial conditions?

Step 1: Write the characteristic equation r^2 + 2r + 5 = 0. Step 2: Find roots r = -1 ± 2i. Step 3: Write the general solution y(t) = e^{-t}(C_1 cos 2t + C_2 sin 2t). Step 4: Use initial conditions y(0) = 3 and y'(0) = 5 to solve for C_1 and C_2. Step 5: Substitute back to obtain the specific solution.

Additional Resources

Solving homogeneous second-order initial value problems (IVPs) is a fundamental aspect of differential equations, with profound implications across physics, engineering, and applied mathematics. Specifically, the differential equation $y'' + 2 y' + 5 y = 0$, coupled with initial conditions $y(0) = 3$ and $y'(0) = 5$, exemplifies a classic second-order linear homogeneous differential equation whose solution process illustrates core analytical techniques. This article provides a comprehensive, detailed analysis of solving such problems, emphasizing the methods, reasoning, and interpretation essential for understanding and applying these solutions in various contexts.

Understanding the Homogeneous Second-Order Differential Equation

Definition and Significance

A second-order differential equation involves the second derivative of an unknown function $y(t)$:

$$
y'' + a y' + b y = 0
$$

where a and b are constants. When the right-hand side is zero, the equation is termed homogeneous. Such equations model systems where the response is solely determined by initial conditions and inherent properties, without external forcing.

In our case:

$$
y'' + 2 y' + 5 y = 0

\]
contains constant coefficients, making it a constant coefficient linear homogeneous differential equation. This form is ubiquitous in systems exhibiting oscillatory or exponential behavior, such as mechanical vibrations, electrical circuits, and quantum mechanics.

Initial Conditions and Their Role

The initial conditions specify the state of the system at a particular point (here, \(t=0 \)):
\[
y(0) = 3, \quad y'(0) = 5
\]
These conditions are essential for determining the unique solution from the general solution of the differential equation. They allow us to solve for arbitrary constants arising during integration, ensuring the solution precisely models the specific initial state.

Methodology for Solving the Differential Equation

Characteristic Equation Approach

The standard method for solving such equations involves seeking solutions of the form \(y(t) = e^{r t} \). Substituting into the differential equation yields the characteristic equation:
\[
r^2 + 2 r + 5 = 0
\]
This quadratic equation determines the roots \(r \), which guide the form of the general solution.

Step-by-Step Solution Strategy

1. Formulate the characteristic equation.
2. Solve for roots \(r \).
3. Determine the general solution based on root types (real or complex).
4. Apply initial conditions to find particular constants.
5. Write the explicit solution.

This systematic approach ensures a rigorous and clear pathway from the differential equation to its specific solution.

Detailed Solution to the Homogeneous Equation

Formulating the Characteristic Equation

Starting with the differential equation:
$$
y'' + 2 y' + 5 y = 0
$$
assume a solution of the form $y(t) = e^{r t}$. Plugging into the equation:
$$
r^2 e^{r t} + 2 r e^{r t} + 5 e^{r t} = 0
$$
Divide through by $e^{r t}$ (which is never zero):
$$
r^2 + 2 r + 5 = 0
$$
This quadratic characteristic equation encapsulates the behavior of solutions.

Solving the Characteristic Equation

Applying the quadratic formula:
$$
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
$$
with $a=1$, $b=2$, $c=5$:
$$
r = \frac{-2 \pm \sqrt{(2)^2 - 4 \times 1 \times 5}}{2}
= \frac{-2 \pm \sqrt{4 - 20}}{2}
= \frac{-2 \pm \sqrt{-16}}{2}
$$
$$
r = \frac{-2 \pm 4 i}{2}
$$
which simplifies to:
$$
r = -1 \pm 2 i
$$

Root Analysis:
- Roots are complex conjugates with real part -1 and imaginary part ± 2.

Constructing the General Solution

For roots $r = \alpha \pm \beta i$, the general solution is:
$$
y(t) = e^{\alpha t} \left(C_1 \cos \beta t + C_2 \sin \beta t \right)
$$
In our case:
$$
\alpha = -1, \quad \beta = 2
$$

thus,

$$
\boxed{
y(t) = e^{-t} \left(C_1 \cos 2 t + C_2 \sin 2 t \right)
}
$$

where C_1 and C_2 are arbitrary constants to be determined by the initial conditions.

Applying Initial Conditions to Find Particular Solution

Initial Condition $y(0) = 3$

Evaluate $y(t)$ at $t=0$:

$$
y(0) = e^{0} (C_1 \cos 0 + C_2 \sin 0) = C_1 \times 1 + C_2 \times 0 = C_1
$$

Hence,

$$
C_1 = 3
$$

Initial Condition $y'(0) = 5$

First, differentiate $y(t)$:

$$
y(t) = e^{-t} (C_1 \cos 2 t + C_2 \sin 2 t)
$$

Applying the product rule:

$$
y'(t) = \frac{d}{dt} [e^{-t}] \times (C_1 \cos 2 t + C_2 \sin 2 t) + e^{-t} \times \frac{d}{dt} [C_1 \cos 2 t + C_2 \sin 2 t]
$$

Calculate derivatives:

$$
\frac{d}{dt} e^{-t} = - e^{-t}
$$

$$
\frac{d}{dt} [C_1 \cos 2 t + C_2 \sin 2 t] = -2 C_1 \sin 2 t + 2 C_2 \cos 2 t
$$

Putting it together:

$$
y'(t) = - e^{-t} (C_1 \cos 2 t + C_2 \sin 2 t) + e^{-t} (-2 C_1 \sin 2 t + 2 C_2 \cos 2 t)
$$

At $t=0$:

$$
y'(0) = - (C_1 \times 1 + C_2 \times 0) + (-2 C_1 \times 0 + 2 C_2 \times 1) = - C_1 + 2 C_2
$$

Substitute known $C_1 = 3$:

$$
5 = - 3 + 2 C_2
$$

Solve for C_2:

$$
2 C_2 = 8 \Rightarrow C_2 = 4
$$

Final Solution and Interpretation

With $C_1 = 3$ and $C_2 = 4$, the specific solution to the initial value problem is:

$$
\boxed{
y(t) = e^{-t} \left(3 \cos 2 t + 4 \sin 2 t \right)
}
$$

Interpretation:
- The exponential decay factor e^{-t} indicates the system's oscillations diminish over time, reflecting damping behavior.
- The oscillatory components $\cos 2 t$ and $\sin 2 t$ demonstrate the system's natural frequency of oscillation, which is $\frac{\beta}{2\pi}$, here related to the imaginary part of roots.
- The initial conditions shape the amplitude and phase of these oscillations, illustrating how initial position and velocity influence long-term behavior.

Broader Context and Applications

Physical Systems Modeled by Such Equations

This mathematical paradigm mirrors many real-world systems:
- Mechanical oscillators with damping (like a spring-mass system with friction).
- Electrical RLC circuits where current or voltage oscillates with resistance.
- Quantum systems where wave functions exhibit exponential decay combined with oscillations.

Understanding how to solve these equations enables engineers and scientists to predict system responses, optimize designs, and interpret experimental data.

Extensions and Variations

While this article focused on homogeneous equations with constant coefficients, many systems involve:
- Nonhomogeneous terms, leading to particular solutions.
- Variable coefficients, requiring special functions or numerical methods.
- Higher-order equations, expanding complexity.

Mastery of the homogeneous second-order case provides a foundation for tackling these advanced challenges.

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Solve The Homogeneous Second Order Initial Value Problem A Y 2y 5y 0 Y 0 3 Y 0 5 B

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