Suppose That The Only Currency Was 3-dollar Bills And 10-dollar Bills. Show That Every Amount Greater

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Introduction

In the world of mathematics and number theory, problems involving currency denominations are classic examples used to illustrate concepts such as linear combinations, the Frobenius coin problem, and the idea of representing integers as sums of given denominations. A particularly interesting problem is to determine which amounts of money can be formed using only certain fixed denominations.

In this article, we explore the scenario where the only currency denominations are 3-dollar bills and 10-dollar bills. Our goal is to demonstrate that every amount greater than or equal to a specific value can be expressed as a combination of these bills. We will delve into the underlying mathematical principles, provide proofs, and discuss the significance of these results in number theory and practical applications.

Understanding the Problem

The Scenario

Suppose that the only currency in circulation is:

- 3-dollar bills
- 10-dollar bills

The central question is:

Can every amount of money greater than a certain threshold be represented as a sum of some number of 3-dollar and 10-dollar bills?

Formal Restatement

Formally, for any integer amount N, is there a non-negative integer solution (x, y) to the equation:

$$
3x + 10y = N
$$

where:

- $x, y \geq 0$
- N is an integer amount greater than or equal to some minimum value.

Our task is to identify the smallest amount N such that all amounts greater than or equal to N can be represented in this form.

Mathematical Foundations

The Frobenius Coin Problem

The problem of representing integers as linear combinations of two positive integers is closely related to the Frobenius coin problem (also known as the Chicken McNuggets theorem for two numbers).

Key Points:

- For two positive, coprime integers a and b, the largest integer that cannot be expressed as a non-negative combination of a and b is:

$$
ab - a - b
$$

- Every integer greater than this value can be expressed as a combination of a and b.

Applying to Our Denominations

In our case:

- $a = 3$
- $b = 10$

Since 3 and 10 are coprime ($\gcd(3, 10) = 1$), the Frobenius coin problem applies directly.

Calculating the Frobenius number:

$$
ab - a - b = (3)(10) - 3 - 10 = 30 - 3 - 10 = 17
$$

Interpretation:

- The largest amount not representable as a non-negative combination of 3 and 10 is 17 dollars.
- All amounts greater than 17 dollars can be expressed as some combination of 3-dollar and 10-dollar bills.

Demonstrating That Every Amount Greater Than 17 Can Be Represented

Step 1: Establish the Base Cases

Let's verify that certain amounts can be represented:

- $N = 18$:

$$
3 \times 6 + 10 \times 0 = 18
$$

- $N = 19$:

$$
3 \times 3 + 10 \times 1 = 9 + 10 = 19
$$

- $N = 20$:

$$
3 \times 0 + 10 \times 2 = 20
$$

- $N = 21$:

$$
3 \times 7 + 10 \times 0 = 21
$$

- $N = 22$:

$$
3 \times 2 + 10 \times 2 = 6 + 20 = 26 \quad \text{(exceeds 22)}
$$

But we need to find a representation for 22:

$$
3 \times 4 + 10 \times 1 = 12 + 10 = 22
$$

Similarly, for 23:

$$
3 \times 1 + 10 \times 2 = 3 + 20 = 23
$$

And for 24:

$$
3 \times 4 + 10 \times 1 = 12 + 10 = 22 \quad \text{(not 24)}
$$

Alternatively:

$$
3 \times 8 + 10 \times 0 = 24
$$

Thus, we see that starting from 18, all amounts up to 24 are representable.

Step 2: Show that All Amounts ≥ 18 are Representable

From the base cases, we observe that:

- 18, 19, 20, 21, 22, 23, 24 are all representable.

Now, observe that:

- For any amount $N \geq 18$, subtracting 3 repeatedly reduces the amount to a value that we have already confirmed is representable.

Step 3: General Proof

Since 3 and 10 are coprime, and the Frobenius number is 17, the following holds:

- Every integer $N \geq 18$ can be expressed as:

$$
N = 3k + 10m
$$

for some non-negative integers k, m.

Proof via Induction:

- Base cases: Confirmed for $N = 18, 19, 20, 21, 22, 23, 24$.

- Inductive step:

Suppose that all amounts N' with $N' \geq 18$ and $N' < N$ are representable.

Now, consider $N \geq 18$.

- Since $N - 3 \geq 15$, which is less than N but still ≥ 18, and we want a more general proof.

Alternatively, note that:

- For any $N \geq 18$, subtract 3:

$$
N - 3 \geq 15
$$

- But 15 is representable (e.g., 3×5), so:

$$
N = (N - 3) + 3
$$

- If $N - 3$ is representable, then adding a 3-dollar bill gives a representation for N.

- Since $N - 3 \geq 15$, and all numbers ≥ 18 can be built from previous cases, the inductive process continues.

Conclusion:

- All amounts $N \geq 18$ can be expressed as a sum of 3-dollar and 10-dollar bills.

- The only amounts less than 18 that cannot be expressed are 1, 2, 4, 5, 7, 8, 11, 13, 14, 17 (verified by direct checking).

Practical Implications

Commercial and Financial Relevance

Understanding which amounts can be made with specific denominations has practical implications:

- Cash transactions: Knowing the minimal amount beyond which any sum can be made with given bills helps in making change efficiently.
- Banking and ATM design: Ensuring the availability of denominations that can cover all amounts beyond a threshold simplifies cash dispensing.
- Currency design: Choosing denominations that are coprime and cover all larger amounts ensures flexibility in transactions.

Mathematical Significance

- Demonstrates the application of the Frobenius coin problem.
- Highlights how number theory can solve real-world problems involving combinatorics and integer representations.
- Serves as an example in teaching linear Diophantine equations and their solutions.

Summary of Key Points

- The only denominations are 3-dollar and 10-dollar bills.
- Since 3 and 10 are coprime, the Frobenius number is $(3)(10) - 3 - 10 = 17$.
- All amounts $N \geq 18$ can be represented as:

$$
N = 3x + 10y, \quad x, y \geq 0
$$

- The amounts less than 18 that cannot be represented are precisely:

$$
1, 2, 4, 5, 7, 8, 11, 13, 14, 17
$$

- This result guarantees that beyond a certain threshold, every amount can be formed using only 3-dollar and 10-dollar bills.

Additional Insights

Extending to Other Denominations

- If the denominations are not coprime, some amounts may never be representable.
- The Frobenius number only applies when the denominations are coprime.
- For multiple denominations, finding all representable amounts becomes more complex.

Limitations and Assumptions

- The proof assumes unlimited availability of bills.
- It assumes non-negative integers for the number of bills.
- Real-world constraints like limited supply or denominations not being integers are not considered.

Final Remarks

The problem of representing amounts with fixed denominations showcases powerful concepts in number theory, especially the Frobenius coin problem. By understanding the properties of coprime integers and their linear combinations, we see that beyond a specific threshold (17 dollars in this case

Frequently Asked Questions

How can we prove that any amount greater than zero can be formed using only 3-dollar and 10-dollar bills?

By using the concept of linear combinations, we can show that for any amount $n \geq 8$, there exist non-negative integers x and y such that $3x + 10y = n$. Specifically, since 3 and 10 are coprime, every sufficiently large amount can be expressed as a combination of these bills.

What is the smallest amount that cannot be formed using only 3-dollar and 10-dollar bills?

The Frobenius number for 3 and 10 is 17, meaning that 17 cannot be formed with any combination of 3 and 10 dollar bills, but all amounts greater than 17 can be.

Can you provide an example of forming an amount greater than 17 using only 3-dollar and 10-dollar bills?

Yes. For example, to make $23, use two 10-dollar bills (20) and one 3-dollar bill (3): 10 + 10 + 3 = 23.

Is there a general formula to determine whether a certain amount can be formed using only 3-dollar and 10-dollar bills?

Yes. Since 3 and 10 are coprime, any amount $n \geq 8$ can be expressed as $3x + 10y$, where x and y are non-negative integers, following the linear Diophantine equation approach.

Why is 17 the largest amount that cannot be formed using 3 and 10-dollar bills?

Because 17 is the Frobenius number for 3 and 10, representing the largest integer that cannot be expressed as a non-negative combination of these two denominations.

How does the concept of the Frobenius number relate to this problem?

The Frobenius number indicates the largest amount that cannot be obtained using combinations of two coprime denominations—in this case, 3 and 10 dollars. All amounts greater than this number can be formed.

Can every amount greater than 17 be formed using only 3-dollar and 10-dollar bills?

Yes. Since 3 and 10 are coprime, and 17 is the Frobenius number, all amounts greater than 17 can be expressed as a linear combination of these bills.

What is the significance of coprimality between 3 and 10 in this context?

Because 3 and 10 are coprime, it guarantees that beyond a certain point (the Frobenius number), all larger amounts can be formed using these denominations.

How can we systematically find the combinations for amounts

greater than 17?

By solving the linear Diophantine equation 3x + 10y = n for non-negative integers x and y, and applying known algorithms or algorithms like the Chicken McNuggets theorem, we can find the combinations for any amount above 17.

Additional Resources

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Introduction: A Puzzle of Currency and Mathematics

Imagine a world where the only forms of currency are 3-dollar bills and 10-dollar bills. It might seem like a simple setup at first glance, but beneath this scenario lies a fascinating mathematical problem—one that touches on concepts of number theory, linear combinations, and the famous “Coin Problem.” This thought experiment is not only intriguing for mathematicians and economists alike but also offers a glimpse into how simple rules can determine the possibilities of making change for any amount of money.

In this article, we will explore whether it is possible to form every amount greater than a certain value using only 3-dollar and 10-dollar bills. We will delve into the core ideas behind the problem, analyze the conditions under which all large amounts can be formed, and demonstrate the general principles that govern such combinations. This journey will reveal how basic arithmetic and number theory provide powerful tools to understand real-world issues like currency denominations and making change.

The Core Question: When Can All Amounts Be Made?

At the heart of this problem is a fundamental question:

Given only 3-dollar and 10-dollar bills, can every amount of money greater than some number be formed?

More precisely, the question asks us to identify the smallest amount beyond which every larger amount can be expressed as a sum of 3-dollar and 10-dollar bills. This is a classic problem in number theory often related to the “Coin Problem” or the “Frobenius Number” problem, which deals with the largest amount that cannot be formed using two coin denominations.

Understanding the Mathematical Framework

Linear Combinations and the Frobenius Number

Suppose we have two positive integers, \(a\) and \(b\), representing the denominations—in our case, 3 and 10. The question becomes: which positive integers N can be expressed as

$$
N = 3x + 10y,
$$

where x and y are non-negative integers (since bills are valued positively and cannot be negative)?

This is known as a linear Diophantine equation, and the set of all amounts that can be formed is called the set of non-negative linear combinations of the denominations.

The key to understanding the problem lies in the Greatest Common Divisor (gcd) of the two denominations:

$$
\gcd(3, 10) = 1.
$$

Because 3 and 10 are coprime (they share no common divisors other than 1), there exists a finite set of amounts that cannot be expressed as a combination of these two bills, but beyond a certain point, all larger amounts can be formed.

The Frobenius Number: The Largest Unreachable Amount

The Frobenius Number, denoted by $g(a, b)$, is the largest integer that cannot be expressed as a non-negative linear combination of a and b.

For two coprime positive integers, the Frobenius Number is given by the formula:

$$
g(a, b) = ab - a - b.
$$

Applying this to our case:

$$
g(3, 10) = (3)(10) - 3 - 10 = 30 - 3 - 10 = 17.
$$

This means:

- The largest amount that cannot be formed using only 3-dollar and 10-dollar bills is 17 dollars.
- All amounts greater than 17 dollars can be expressed as a sum of 3-dollar and 10-dollar bills.

In other words: every amount greater than 17 dollars can be made using some combination of 3-dollar and 10-dollar bills.

Demonstrating That All Amounts Greater Than 17 Can Be Formed

Having established the Frobenius Number as 17, the next step is to prove that every amount exceeding 17 can indeed be constructed from these bills.

Step 1: Confirm the Non-Representable Amounts

First, identify which amounts less than or equal to 17 cannot be formed:

- 1 dollar: cannot be formed (less than 3).
- 2 dollars: cannot be formed.
- 3 dollars: yes (one 3-dollar bill).
- 4 dollars: cannot be formed.
- 5 dollars: cannot be formed.
- 6 dollars: yes (two 3-dollar bills).
- 7 dollars: cannot be formed.
- 8 dollars: cannot be formed.
- 9 dollars: yes (three 3-dollar bills).
- 10 dollars: yes (one 10-dollar bill).
- 11 dollars: yes (one 10 + one 3).
- 12 dollars: yes (four 3-dollar bills).
- 13 dollars: yes (one 10 + one 3).
- 14 dollars: yes (one 10 + two 3's).
- 15 dollars: yes (five 3's).
- 16 dollars: yes (one 10 + two 3's).
- 17 dollars: cannot be formed.

From this, the only amounts less than or equal to 17 that cannot be formed are 1, 2, 4, 5, 7, 8, and 17.

Step 2: Show That All Amounts Greater Than 17 Can Be Formed

Now, focus on amounts greater than 17. For each amount $N > 17$, we want to demonstrate it can be expressed as $3x + 10y$.

Since the Frobenius number is 17, and because 3 and 10 are coprime, the key property is:

> For all integers $N \geq g(3,10) + 1 = 18$, N can be expressed as $3x + 10y$.

This is a standard result in the theory of the Frobenius Number, but we can also verify it through constructive methods.

Constructive Approach: Building Larger Amounts

Let's verify this with some examples:

- 18 dollars: $18 = 3 \times 6$.
- 19 dollars: $19 = 10 + 3 \times 3$.
- 20 dollars: $20 = 10 \times 2$.

- 21 dollars: $21 = 3 \times 7$.
- 22 dollars: $22 = 10 + 3 \times 4$.
- 23 dollars: $23 = 10 + 10 + 3 \times 1$.
- 24 dollars: $24 = 3 \times 8$.
- 25 dollars: $25 = 10 + 3 \times 5$.

From these examples, it's clear that for any amount beyond 17, we can subtract a multiple of 10 and be left with an amount divisible by 3.

More generally, for any $N > 17$:

$$
N - 10k \quad \text{must be divisible by 3,}
$$

for some integer $k \geq 1$. Since 10 mod 3 is 1, subtracting 10 repeatedly shifts the remainder by 1 each time, and because 3 divides all multiples of 3, eventually, for sufficiently large N, the remainder will be divisible by 3, confirming the possibility of representation.

Formal Proof of the Coverage

The previous examples illustrate the general principle. Formally:

- For any $N > 17$, choose k such that:

$$
N - 10k \geq 0,
$$

and

$$
N - 10k \equiv 0 \ (\text{mod } 3).
$$

Given that $10 \equiv 1 \ (\text{mod } 3)$, we have:

$$
N - 10k \equiv N - k \equiv 0 \ (\text{mod } 3),
$$

which implies:

$$
N \equiv k \ (\text{mod } 3).
$$

Since $N > 17$, and the possible residues modulo 3 are 0, 1, or 2, for large enough N, there exists an integer k satisfying these conditions. After determining the suitable k, the remaining amount

$N - 10k$ is divisible by 3, and thus can be formed by some number of 3-dollar bills.

Implications: The Coin Problem in Action

The analysis of this problem illustrates a fundamental principle in the "Coin Problem" or Frobenius Number theory:

- When you have two denominations a and b that are coprime, the largest amount that cannot be formed is $ab - a - b$.
- All larger amounts can be formed by some combination of these coins.

In the context of our currency scenario:

- The largest amount that cannot be formed with 3-dollar and 10-dollar bills is 17 dollars.
- Every amount greater than 17 dollars can be constructed using some non-negative combination of these bills.

This has practical implications, especially for cashiers and financial planners, because it means that beyond a certain threshold, making exact change becomes straightforward, regardless of the specific amounts involved.

Broader Perspectives and Applications

While the original problem is

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