Suppose Utility Is Given By U(x,y)=x 0.1 Y 0.3 : (a) Using A Legrangian, Solve For The Optimal Values

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Introduction to Utility Maximization and the Lagrangian Method

Understanding consumer choice involves analyzing how individuals allocate their limited resources across various goods to maximize their satisfaction or utility. When a consumer faces a budget constraint, the problem becomes an optimization task: maximize utility subject to the budget constraint.

The Lagrangian method is a powerful technique in constrained optimization problems. It introduces a Lagrange multiplier to incorporate the constraint into the objective function, allowing us to find critical points that satisfy both the utility maximization and the budget restriction.

In this article, we will explore how to solve a utility maximization problem where the utility function is given by $U(x, y) = x^{0.1} y^{0.3}$, subject to a budget constraint. We will formulate the problem, set up the Lagrangian, derive the first-order conditions, and solve for the optimal consumption bundle.

Setup of the Utility Maximization Problem

Utility Function

The utility function in question is:

$$
U(x, y) = x^{0.1} y^{0.3}
$$

where:

- x and y are quantities of two goods,
- The exponents reflect the marginal contributions of each good to overall utility.

Budget Constraint

Suppose the consumer has a fixed income M and prices p_x and p_y for goods x and y respectively. The budget constraint is:

$$
p_x x + p_y y = M
$$

Our goal is to determine the optimal quantities $x^$ and $y^$ that maximize utility given this constraint.

Formulating the Lagrangian

The Lagrangian function incorporates the utility function and the budget constraint via a Lagrange multiplier λ:

$$
\mathcal{L}(x, y, \lambda) = x^{0.1} y^{0.3} + \lambda (M - p_x x - p_y y)
$$

This formulation allows us to find stationary points where the gradient of the Lagrangian with respect to x, y, and λ is zero.

Deriving the First-Order Conditions (FOCs)

To identify the optimal bundle, differentiate the Lagrangian with respect to each variable and set the derivatives equal to zero:

Partial derivatives with respect to x and y:

$$
\frac{\partial \mathcal{L}}{\partial x} = 0.1 x^{-0.9} y^{0.3} - \lambda p_x = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial y} = 0.3 x^{0.1} y^{-0.7} - \lambda p_y = 0
$$

Partial derivative with respect to \(\lambda \):

\[
\frac{\partial \mathcal{L}}{\partial \lambda} = M - p_x x - p_y y = 0
\]

These equations form a system that can be solved simultaneously to find \(x^ \), \(y^ \), and \(\lambda \).

Solving the First-Order Conditions

Step 1: Express \(\lambda \) from the first-order conditions

From the derivatives, we get:

\[
\lambda = \frac{0.1 x^{-0.9} y^{0.3}}{p_x}
\]
\[
\lambda = \frac{0.3 x^{0.1} y^{-0.7}}{p_y}
\]

Since both expressions are equal to \(\lambda \), set them equal to each other:
\[
\frac{0.1 x^{-0.9} y^{0.3}}{p_x} = \frac{0.3 x^{0.1} y^{-0.7}}{p_y}
\]

Step 2: Simplify to find the ratio \(y/x \)

Cross-multiplied:
\[
0.1 x^{-0.9} y^{0.3} p_y = 0.3 x^{0.1} y^{-0.7} p_x
\]

Divide both sides by \(x^{-0.9} y^{-0.7} \):
\[
0.1 p_y y^{0.3 + 0.7} = 0.3 p_x x^{0.1 + 0.9}
\]

Simplify exponents:
\[
0.1 p_y y^{1.0} = 0.3 p_x x^{1.0}
\]

Express \(y \) in terms of \(x \):
\[
0.1 p_y y = 0.3 p_x x
\]
\[
p_y y = 3 p_x x
\]
\[
y = \frac{3 p_x}{p_y} x
\]

This relation indicates the optimal ratio of \(y \) to \(x \) depends on the prices.

Step 3: Use the budget constraint to solve for \(x \) and \(y \)

Substitute \(y \) into the budget constraint:
\[
p_x x + p_y y = M
\]
\[
p_x x + p_y \left(\frac{3 p_x}{p_y} x \right) = M
\]
\[
p_x x + 3 p_x x = M
\]
\[
4 p_x x = M
\]
\[
x^ = \frac{M}{4 p_x}
\]

Now, find \(y^ \):
\[
y^ = \frac{3 p_x}{p_y} x^ = \frac{3 p_x}{p_y} \times \frac{M}{4 p_x} = \frac{3 M}{4 p_y}
\]

Final Results for the Optimal Consumption Bundle

The optimal quantities are:

- For good \(x \):
  \[
  x^ = \frac{M}{4 p_x}
  \]

- For good \(y \):
  \[
  y^ = \frac{3 M}{4 p_y}
  \]

These solutions reflect the consumer's optimal consumption given their income, prices, and the utility function.

Interpretation and Insights

Budget Allocation

The consumer allocates their budget proportionally:
- One-quarter (\(\frac{1}{4} \)) of their income spent on good \(x \),
- Three-quarters (\(\frac{3}{4} \)) on good \(y \).

This allocation aligns with the exponents in the utility function, indicating the relative importance of each good to the consumer's utility.

Impact of Prices and Income

- As \(M \) increases, both \(x^ \) and \(y^ \) increase proportionally.
- Changes in prices \(p_x \) and \(p_y \) influence the optimal quantities inversely, consistent with the law of demand.

Role of the Utility Exponents

The exponents \(0.1 \) and \(0.3 \) determine the consumer’s marginal valuations:
- The larger exponent for \(y \) suggests \(y \) provides relatively more utility per unit increase than \(x \),
- The ratios derived from the exponents reflect these preferences.

Conclusion

Using the Lagrangian method, we successfully derived the consumer's optimal consumption bundle for the utility function $U(x, y) = x^{0.1} y^{0.3}$. The key steps involved setting up the Lagrangian, deriving the first-order conditions, equating the marginal utility per dollar spent for both goods, and solving the resulting equations under the budget constraint.

The approach illustrates how the mathematical framework of constrained optimization provides clear insights into consumer behavior, revealing how income, prices, and preferences shape optimal consumption choices. This methodology is foundational in microeconomic theory, underpinning analyses of demand, market equilibrium, and welfare economics.

Note: If specific values for M, p_x, and p_y are provided, the numerical optimal quantities can be directly computed using the formulas derived.

Frequently Asked Questions

What is the main goal when solving for the optimal consumption bundle given the utility function U(x,y) = x^{0.1} y^{0.3}?

The main goal is to maximize the utility function subject to the consumer's budget constraint, finding the optimal amounts of x and y that provide the highest utility within the available income and prices.

How do you set up the Lagrangian for the utility maximization problem with the utility U(x,y) = x^{0.1} y^{0.3}?

The Lagrangian is set up as $L = x^{0.1} y^{0.3} + \lambda (I - p_x x - p_y y)$, where I is income, p_x and p_y are the prices of x and y respectively.

What are the first-order conditions (FOCs) derived from the Lagrangian in this problem?

The FOCs are: $\partial L/\partial x = 0.1 x^{-0.9} y^{0.3} - \lambda p_x = 0$; $\partial L/\partial y = 0.3 x^{0.1} y^{-0.7} - \lambda p_y = 0$; and the budget constraint $p_x x + p_y y = I$.

How do you solve for the optimal ratio of x to y using the first-order conditions?

Dividing the first FOC by the second eliminates λ, leading to (0.1 / 0.3) ($x^{-0.9} y^{0.3}$) / ($x^{0.1} y^{-0.7}$) = p_x / p_y, which simplifies to a ratio of x and y in terms of prices.

What is the expression for the optimal consumption bundle (x, y) after solving the FOCs?

The optimal values are obtained by solving the ratio condition along with the budget constraint, resulting in $x = [(\alpha p_y) / (\beta p_x)]^{1 / (\alpha + \beta)} (I / (p_x + p_y))$, and similarly for y, where $\alpha=0.1$ and $\beta=0.3$.

How does the Cobb-Douglas form of the utility function influence the solution for the optimal consumption bundle?

Since the utility function is Cobb-Douglas, the optimal consumption proportions are constant and proportional to the exponents, and the solution involves allocating income based on these exponents and prices.

What role does the Lagrange multiplier λ play in the solution process?

The multiplier λ represents the marginal utility of income and helps relate the marginal utilities of x and y to their prices, ensuring the consumer maximizes utility within their budget constraint.

Can the approach used for solving this utility maximization problem be generalized to other utility functions?

Yes, setting up the Lagrangian and deriving FOCs is a general method applicable to various utility functions, though the specific solutions depend on the functional form and constraints involved.

Additional Resources

Suppose Utility Is Given By $U(x,y)=x^{0.1} y^{0.3}$: (a) Using A Lagrangian, Solve For The Optimal Values

In the realm of consumer theory and microeconomic analysis, understanding how individuals allocate their limited resources to maximize satisfaction is fundamental. Mathematically, this involves solving optimization problems where a utility function—reflecting preferences—is maximized subject to budget constraints. Today, we explore such a problem with a specific utility function: $U(x, y) = x^{0.1} y^{0.3}$. Our goal is to determine the optimal quantities of goods x and y that maximize utility given a budget constraint, employing the powerful method of Lagrangian multipliers. This process not only highlights the mathematical elegance of constrained optimization but also offers insights into consumer behavior.

Understanding the Utility Function and Constraints

Before diving into the solution, it's essential to interpret the utility function and the problem setting:

- Utility Function: $U(x, y) = x^{0.1} y^{0.3}$
This Cobb-Douglas form signifies how consumers derive satisfaction from consuming quantities of goods x and y. The exponents (0.1 for x and 0.3 for y) indicate the relative importance or preference strength for each good.

- Budget Constraint: Suppose the consumer has a total income, denoted by M, and prices p_x and p_y for goods x and y respectively. The budget constraint is:
$p_x x + p_y y = M$

The objective is to select quantities x and y that maximize U(x, y) while satisfying this budget constraint.

Setting Up the Optimization Problem

The formal problem statement is:

Maximize $U(x,y) = x^{0.1} y^{0.3}$
Subject to $p_x x + p_y y = M$

This is a constrained optimization problem, which can be tackled with the method of Lagrange multipliers.

Applying the Lagrangian Method

The Lagrangian function integrates the objective function with the constraint, introducing a multiplier λ (lambda):

$L(x, y, \lambda) = x^{0.1} y^{0.3} + \lambda (M - p_x x - p_y y)$

This function allows us to find stationary points where the partial derivatives with respect to x, y, and λ are zero.

Step 1: Derive the First-Order Conditions

To locate the optimum, compute the partial derivatives:

1. Derivative with respect to x:

$$\partial L/\partial x = 0.1 x^{-0.9} y^{0.3} - \lambda p_x = 0$$

2. Derivative with respect to y:

$$\partial L/\partial y = 0.3 x^{0.1} y^{-0.7} - \lambda p_y = 0$$

3. Derivative with respect to λ:

$$\partial L/\partial \lambda = M - p_x x - p_y y = 0$$

These conditions form a system of equations to solve for x, y, and λ.

Step 2: Express λ from the First-Order Conditions

From the first two equations, isolate λ:

- From $\partial L/\partial x$:

$$\lambda = (0.1 x^{-0.9} y^{0.3}) / p_x$$

- From $\partial L/\partial y$:

$$\lambda = (0.3 x^{0.1} y^{-0.7}) / p_y$$

Since both expressions equal λ, set them equal to each other:

$$(0.1 x^{-0.9} y^{0.3}) / p_x = (0.3 x^{0.1} y^{-0.7}) / p_y$$

Step 3: Simplify to Find the Relationship Between x and y

Cross-multiplied, the equation becomes:

$$0.1 x^{-0.9} y^{0.3} p_y = 0.3 x^{0.1} y^{-0.7} p_x$$

Divide both sides by common terms:

$$(0.1 / 0.3) (p_y / p_x) = x^{0.1 + 0.9} y^{-0.7 - 0.3}$$

Note that:

- x exponents: -0.9 and 0.1 sum to 0.1 + 0.9 = 1

- y exponents: 0.3 and -0.7 sum to -0.4

Thus, the equation simplifies to:

$$(1/3) (p_y / p_x) = x^{1} y^{-0.4}$$

Expressed as:

$$x y^{-0.4} = (1/3) (p_y / p_x)$$

Or equivalently:

$$x = (1/3) (p_y / p_x) y^{0.4}$$

This relationship links x and y, providing a basis for calculating their optimal amounts.

Solving for the Optimal Quantities

Using the relation derived, the next step involves substituting back into the budget constraint to determine the explicit values of x and y.

Step 4: Substitute into the Budget Constraint

Recall:

$$p_x x + p_y y = M$$

Substitute x:

$$p_x [(1/3) (p_y / p_x) y^{0.4}] + p_y y = M$$

Simplify:

$$(1/3) p_x (p_y / p_x) y^{0.4} + p_y y = M$$

The p_x terms cancel in the first part:

$$(1/3) p_y y^{0.4} + p_y y = M$$

Factor out p_y:

$$p_y [(1/3) y^{0.4} + y] = M$$

Now, solve for y:

$$p_y [(1/3) y^{0.4} + y] = M$$

$$\Rightarrow ((1/3) y^{0.4} + y) = M / p_y$$

Define $S(y)$:

$$S(y) = (1/3) y^{0.4} + y$$

Set $S(y) = M / p_y$

This is a nonlinear equation in y, which generally requires numerical methods for exact solutions. However, understanding its structure provides valuable insights into how the budget and prices influence optimal consumption.

Step 5: Find y and then x

Once y is obtained from solving $S(y) = M / p_y$, substitute back into the expression for x:

$$x = (1/3) (p_y / p_x) y^{0.4}$$

This yields the optimal quantities:

- Optimal y:

Solved numerically from the nonlinear equation

- Optimal x:

Computed directly from the relationship with y

Interpreting the Results and Economic Intuition

The mathematical solution reveals several key insights:

- The ratio of marginal utilities to prices determines the optimal consumption bundle.

- The relationship between x and y is influenced by relative prices (p_x and p_y) and income (M).

- Due to the Cobb-Douglas form, the consumer allocates income proportionally to the exponents, adjusted by prices.

Specifically, the exponents (0.1 for x and 0.3 for y) imply that the consumer derives relatively more utility from y, leading to higher consumption of y compared to x, holding prices and income constant.

Conclusion: The Power of the Lagrangian Approach

Using the Lagrangian method to solve the utility maximization problem offers a systematic and elegant way to handle constraints. It transforms a potentially complex problem into manageable equations that reveal the relationships between quantities, prices, and income. Although real-world problems often require numerical methods for precise solutions, the analytical framework provides deep insights into consumer preferences and market behavior.

In this specific example, the utility function's structure, combined with the budget constraint, illustrates how consumers optimize their consumption choices. Whether in academic research or practical policy analysis, mastering the Lagrangian approach remains an essential tool for economists seeking to understand decision-making processes under constraints.

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