Suppose V Is Finite-dimensional And S, T In L(v). Prove That S And T Are Invertible If And Only If St

Suppose V Is Finite-dimensional And S, T In L(v). Prove That S And T Are Invertible If And Only If St

Understanding the conditions under which linear operators are invertible is fundamental in linear algebra, especially when dealing with finite-dimensional vector spaces. In particular, the relationship between the invertibility of operators S and T and the invertibility of their composition St offers deep insights into the structure of linear operators and their invertibility properties. This article explores the equivalence between the invertibility of S and T individually and the invertibility of their composition St, providing a comprehensive proof and discussion in a step-by-step manner.

Preliminaries and Notation

Before delving into the main proof, it's essential to establish some basic definitions and notation used throughout the discussion:

Linear Operators and the Space L(V)

- **V**: A finite-dimensional vector space over a field (usually $\mathbb{R}$ or $\mathbb{C}$).
- **L(V)**: The set of all linear operators (linear transformations) from V to V.

Invertibility of Linear Operators

- A linear operator $S \in L(V)$ is invertible if there exists an operator $S^{-1} \in L(V)$ such that:
  - $S S^{-1} = S^{-1} S = I$, where I is the identity operator on V.
- Equivalent conditions for invertibility include:

- S is bijective (both injective and surjective).
- The matrix representation of S (relative to any basis) is invertible.

The Main Theorem: Invertibility of S and T and Their Composition

The core statement we aim to prove is:

Theorem: Suppose V is finite-dimensional and $S, T \in L(V)$. Then S and T are invertible if and only if the composition ST is invertible.

This theorem establishes a critical link between the invertibility of individual operators and their composition, highlighting the equivalence in the finite-dimensional case.

Proof of the Theorem

The proof proceeds in two directions:

- (⇒) If S and T are invertible, then their composition ST is invertible.
- (⇐) If ST is invertible, then both S and T are invertible.

Let's analyze each part carefully.

Part 1: If S and T are invertible, then ST is invertible

This direction is straightforward.

1. Assume S and T are invertible operators on V.
2. By definition, there exist $S^{-1}, T^{-1} \in L(V)$ such that:
   - $S S^{-1} = S^{-1} S = I$
   - $T T^{-1} = T^{-1} T = I$

3. To show that ST is invertible, consider the operator $T^{-1} S^{-1}$.
4. Compute:
   - $(ST)(T^{-1} S^{-1}) = S(T T^{-1}) S^{-1} = S I S^{-1} = S S^{-1} = I$
   - $(T^{-1} S^{-1})(ST) = T^{-1} (S^{-1} S) T = T^{-1} I T = T^{-1} T = I$
5. Therefore, $(T^{-1} S^{-1})$ is the inverse of ST.
6. Hence, ST is invertible, with inverse $T^{-1} S^{-1}$.

Conclusion: If S and T are invertible, then so is their composition ST.

Part 2: If ST is invertible, then S and T are invertible

This direction is more involved and requires a detailed argument.

1. Assume ST is invertible, i.e., there exists an operator $U \in L(V)$ such that:
   - $U (ST) = (ST) U = I$
2. Our goal is to demonstrate that both S and T are invertible.
3. Start by showing that T is invertible.
4. Observe that:
   - Since ST is invertible, $V = (ST)(V) = V$, so the image of ST is the entire space V.
5. Consider the operator $S: V \to V$. To show S is invertible, we need to demonstrate that S is both injective and surjective.

6. Focus on T first:
   - Suppose T is not invertible. Then, T is either not injective or not surjective.
   - In finite-dimensional spaces, linear operators are invertible if and only if they are bijective.
   - Assuming T is not invertible, then either:
     - $\ker(T) \neq \{0\}$
     - or T is not onto

7. Show that $\ker(T) = \{0\}$:
   - Suppose there exists $v \neq 0$ with $T(v) = 0$.
   - Then:
     - $(ST)(v) = S(T(v)) = S(0) = 0$
   - Since (ST) is invertible, it is injective, so:
     - $(ST)(v) = 0 \Rightarrow v = 0$
   - This contradicts $v \neq 0$, so $\ker(T) = \{0\}$.

8. Show that T is surjective:
   - Given that ST is invertible, its image is V, so for any $w \in V$, there exists $v \in V$ such that:
     - $(ST)(v) = w$
   - Write:
     - $w = (ST)(v) = S(T(v))$

- Since S is linear, and T(v) runs over a subspace of V as v varies, if S is invertible, then T(v) can generate all of V.

9. To establish the invertibility of S, note that:

    - Since ker(T) = {0} and T is injective, T is invertible onto its image.

    - Because T is injective and the space is finite-dimensional, T is surjective if and only if T is invertible.

    - Now, consider (ST) is invertible and the space V has finite dimension. The invertibility of ST implies that the images of T span V, so T is onto, hence invertible.

10. Finally, since T is invertible, and ST is invertible, it follows that S = (ST) T^{-1} is a product of invertible operators, hence S is invertible.

Summary of the argument:

- The invertibility of ST implies T is injective and surjective; thus, T is invertible.
- With T invertible, S = (ST) T^{-1} is a product of invertible operators; hence, S is invertible.

Conclusion: If ST is invertible, then both S and T are invertible.

Implications and Applications

Understanding this theorem has numerous implications in linear algebra and related fields:

Matrix Representation and Similarity

- Since finite-dimensional spaces allow basis choices, the invertibility of linear operators corresponds to invertibility of their matrix representations.

- This theorem reflects that the product of invertible matrices is invertible, and the invertibility of a product matrix implies the invertibility of the factors.

Operator Theory and Functional Analysis

- In more advanced contexts, similar properties hold for bounded operators on Banach spaces, although proofs may require additional considerations.
- This theorem emphasizes the importance of invertibility properties in understanding the structure of operator algebras.

Practical Applications

A product of invertible matrices is invertible, and the inverse of the product is the product of the inverses in reverse order. Conversely, if a product of matrices is invertible, then each matrix must be invertible.

This is a fundamental result in matrix algebra, underpinning many computational procedures and theoretical frameworks.

Extensions and

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