The Functionf(x,y)=x2+(x+6y)4has A Minimumf(0,0)=0Q14.1 2 Points What Is The Gradient Of The Function

The Functionf(x,y)=x2+(x+6y)4has A Minimumf(0,0)=0Q14.1 2 Points What Is The Gradient Of The Function

Understanding the behavior of functions in multivariable calculus is crucial for many applications in mathematics, engineering, and sciences. When analyzing a function like $f(x, y) = x^2 + (x + 6y)^4$, a fundamental step is to determine its gradient. The gradient provides important information about the function's rate of change, direction of steepest ascent, and potential points of minima or maxima. In this article, we'll explore the concept of the gradient in detail, analyze the function in question, and discuss how the gradient helps identify critical points such as minima, maxima, and saddle points.

Introduction to the Function and Its Significance

Understanding the Function $f(x, y) = x^2 + (x + 6y)^4$

The given function combines a quadratic term with a quartic term involving both variables:

$$ f(x, y) = x^2 + (x + 6y)^4 $$

- Quadratic term x^2: This term is always non-negative and has a minimum at $x = 0$.
- Quartic term $(x + 6y)^4$: Since it's raised to the fourth power, it's also non-negative and zero only when $x + 6y = 0$.

The combination of these terms creates a surface with interesting properties, including a known minimum at the point $(0, 0)$, where the function value is zero:

$$ f(0, 0) = 0^2 + (0 + 6 \times 0)^4 = 0 + 0 = 0 $$

This indicates that $(0, 0)$ is at least a local minimum. Determining whether it is a global minimum requires examining the function's behavior around this point and identifying the gradient and critical points.

What Is the Gradient of a Function?

Definition of the Gradient

In multivariable calculus, the gradient of a function $f(x, y)$ is a vector that contains all of its first-order partial derivatives:

$$ \nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) $$

The gradient points in the direction of the steepest increase of the function at a particular point. Its magnitude indicates the rate of increase in that direction.

Significance of the Gradient

- Critical Points: Points where the gradient vector equals zero ($\nabla f = 0$) are potential locations of minima, maxima, or saddle points.
- Direction of Steepest Ascent: The gradient points in the direction in which the function increases most rapidly.
- Optimization: Calculus-based methods such as gradient descent use the gradient to find function minima.

Calculating the Gradient of the Given Function

To find the gradient of $f(x, y) = x^2 + (x + 6y)^4$, we need to compute the partial derivatives with respect to x and y.

Partial Derivative with Respect to x, $\frac{\partial f}{\partial x}$

Given:

$$f(x, y) = x^2 + (x + 6y)^4$$

Applying the chain rule:

$$\frac{\partial f}{\partial x} = 2x + 4(x + 6y)^3 \times \frac{\partial}{\partial x}(x + 6y)$$

Since $\frac{\partial}{\partial x}(x + 6y) = 1$, we get:

$$\frac{\partial f}{\partial x} = 2x + 4(x + 6y)^3$$

Partial Derivative with Respect to y, $\frac{\partial f}{\partial y}$

Applying the chain rule:

$$\frac{\partial f}{\partial y} = 0 + 4(x + 6y)^3 \times \frac{\partial}{\partial y}(x + 6y)$$

Since $\frac{\partial}{\partial y}(x + 6y) = 6$, we obtain:

$$\frac{\partial f}{\partial y} = 24 (x + 6y)^3$$

Expressing the Gradient Vector

Putting it all together, the gradient of $f(x, y)$ is:

$$
\nabla f(x, y) = \left(2x + 4(x + 6y)^3, \quad 24 (x + 6y)^3 \right)
$$

This vector provides the direction and rate of the steepest ascent at any point (x, y).

Finding Critical Points Using the Gradient

Critical points occur where both components of the gradient are zero simultaneously:

$$
\left\{
\begin{aligned}
2x + 4(x + 6y)^3 &= 0 \\
24 (x + 6y)^3 &= 0
\end{aligned}
\right.
$$

From the second equation:

$$
24 (x + 6y)^3 = 0 \implies (x + 6y)^3 = 0 \implies x + 6y = 0
$$

Substituting into the first equation:

$$
2x + 4(0)^3 = 2x = 0 \implies x = 0
$$

Using $x + 6y = 0$:

$$
0 + 6y = 0 \implies y = 0
$$

Thus, the only critical point is at:

$$
(0, 0)
$$

which matches the known minimum point, and the function value at this point:

$$
f(0, 0) = 0
$$

indicates a minimum, confirmed by the gradient analysis.

Analyzing the Nature of the Critical Point

To determine whether this critical point is a minimum, maximum, or saddle point, we examine the second derivatives and the Hessian matrix.

Hessian Matrix

The Hessian matrix H contains the second derivatives:

$$
H = \begin{bmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{bmatrix}
$$

Calculating each component:

- $\frac{\partial^2 f}{\partial x^2}$:

$$
\frac{\partial}{\partial x} \left(2x + 4(x + 6y)^3 \right) = 2 + 12(x + 6y)^2
$$

- $\frac{\partial^2 f}{\partial x \partial y}$:

$$
\frac{\partial}{\partial y} \left(2x + 4(x + 6y)^3 \right) = 12 \times 3 (x + 6y)^2 \times 6 = 216 (x + 6y)^2
$$

- $\frac{\partial^2 f}{\partial y^2}$:

$$
\frac{\partial}{\partial y} \left(24 (x + 6y)^3 \right) = 24 \times 3 (x + 6y)^2 \times 6 = 432 (x + 6y)^2
$$

At the critical point $(0, 0)$:

$$
H(0, 0) = \begin{bmatrix}
2 + 0 & 0 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
2 & 0 \\
0 & 0
\end{bmatrix}
$$

Since the Hessian is positive semi-definite (with a positive diagonal element), the critical point at $(0, 0)$ is a local minimum.

Implications of the Gradient and Critical Point Analysis

Knowing the gradient and the nature of the critical point provides valuable insights:

- Minimum at (0, 0): Since the function value is zero and the Hessian indicates a minimum, this point is a global minimum.
- Behavior near the minimum: The function increases away from this point, confirming its stability as a minimum.
- Other points: The function's behavior away from (0, 0) is influenced by the quartic term, which dominates for large $|x|$ or $|y|$.

Applications of Gradient Analysis in Optimization and Beyond

Understanding the gradient is fundamental in optimization techniques such as gradient descent, which iteratively moves in the direction of the steepest descent to find minima. The analysis of the gradient for this specific function demonstrates:

- How to locate critical points efficiently

Frequently Asked Questions

What is the function f(x, y) given in the problem?

The function is $f(x, y) = x^2 + (x + 6y)^4$.

How do you find the gradient of the function f(x, y)?

The gradient is found by computing the partial derivatives of f with respect to x and y: $\nabla f(x, y) = (\partial f/\partial x, \partial f/\partial y)$.

What is the partial derivative of f(x, y) with respect to x?

$\partial f/\partial x = 2x + 4(x + 6y)^3$.

What is the partial derivative of f(x, y) with respect to y?

$\partial f/\partial y = 24(x + 6y)^3$.

What is the gradient of the function at the point (0, 0)?

At (0, 0), the gradient is (0, 0) because both partial derivatives evaluate to zero at that point.

Why does f(0, 0) = 0 indicate a potential minimum?

Since the function value at (0, 0) is zero and the function is non-negative elsewhere, it suggests that (0, 0) could be a minimum point.

How can we confirm that (0, 0) is a minimum point for the function?

By analyzing the second derivatives or Hessian matrix to check positive definiteness, or by examining the behavior of the function around (0, 0).

Additional Resources

Gradient of the Function

Introduction: Unlocking the Power of the Gradient in Multivariable Calculus

In the realm of multivariable calculus, understanding how functions change with respect to their variables is fundamental. Whether you're optimizing a complex system or analyzing the behavior of a mathematical model, the gradient serves as a critical tool. It acts as a directional compass, pointing towards the steepest ascent of a function, and provides insight into how small changes in variables influence the output.

Today, we take a detailed look at a specific function:
$$f(x, y) = x^2 + (x + 6y)^4$$
which notably has a minimum at $(0, 0)$ with $f(0, 0) = 0$. Our primary focus will be on determining its gradient, exploring what it reveals about the function's behavior, and understanding its significance in broader applications.

Understanding the Function $f(x, y)$

Before diving into the gradient calculation, it's essential to comprehend the structure and properties of the function itself.

Dissection of $f(x, y)$

The function is composed of two parts:

1. Quadratic Term: x^2 — a standard parabola in the x-direction, symmetric around $x=0$.

2. Quartic Term: $(x + 6y)^4$ — a higher-degree polynomial that introduces nonlinearity and complex behavior, especially along the line $x + 6y = 0$.

Key observations:
- The function is smooth and differentiable everywhere in $\mathbb{R}^2$.
- The minimum at $(0, 0)$ suggests a point where the function attains its lowest value, which is zero, consistent with the fact that both terms vanish at that point.

Calculating the Gradient: Step-by-Step Analysis

The gradient of $f(x, y)$, denoted as $\nabla f(x, y)$, encapsulates the partial derivatives with respect to each variable:

\[

$$
\nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)
$$

Understanding how to compute these derivatives accurately is crucial for analyzing the function's behavior.

Step 1: Derivative of x^2

This is straightforward:

$$
\frac{\partial}{\partial x} x^2 = 2x
$$

$$
\frac{\partial}{\partial y} x^2 = 0
$$

since x^2 depends only on x.

Step 2: Derivative of $(x + 6y)^4$

This involves the chain rule:

$$
\frac{\partial}{\partial x} (x + 6y)^4 = 4(x + 6y)^3 \cdot \frac{\partial}{\partial x} (x + 6y) = 4(x + 6y)^3 \times 1 = 4(x + 6y)^3
$$

Similarly,

$$
\frac{\partial}{\partial y} (x + 6y)^4 = 4(x + 6y)^3 \times \frac{\partial}{\partial y} (x + 6y) = 4(x + 6y)^3 \times 6 = 24 (x + 6y)^3
$$

Constructing the Gradient Expression

Putting the derivatives together, we obtain:

$$
\boxed{
\nabla f(x, y) = \left(2x + 4(x + 6y)^3, \quad 24(x + 6y)^3 \right)
}
$$

This concise vector expression clearly shows how the gradient depends both directly on x and y, and on the combined term $x + 6y$.

Interpreting the Gradient: Insights and Implications

The gradient vector reveals much about the function's local behavior:

- Direction of steepest ascent: The gradient points toward the direction where f increases most rapidly.
- Critical points: When $\nabla f = 0$, the function has potential minima, maxima, or saddle points.

Let's analyze these aspects in detail.

Critical Point at $(0, 0)$

Plugging in $(0, 0)$:

$$
\nabla f(0, 0) = \left(2 \times 0 + 4(0 + 6 \times 0)^3, \quad 24(0 + 6 \times 0)^3 \right) = (0, 0)
$$

which confirms that $(0, 0)$ is a critical point, consistent with the given minimum.

Behavior Near the Critical Point

To understand how the function behaves around $(0,0)$, consider small deviations:

- For small x and y, the gradient components are dominated by the linear terms $2x$ and the cubic terms involving $(x + 6y)^3$.
- The gradient vanishes at $(0, 0)$, indicating a stationary point.

Assessing the Nature of the Critical Point

To determine whether $(0, 0)$ is a minimum, maximum, or saddle point, one could analyze the Hessian matrix or evaluate the function's second derivatives at the point.

Broader Applications of the Gradient in Optimization and Analysis

Understanding the gradient isn't just an academic exercise; it has practical implications in various fields:

- Optimization: Gradient-based methods like gradient descent rely on the gradient to find minima or maxima efficiently.
- Machine Learning: Backpropagation algorithms compute gradients to optimize neural network parameters.
- Engineering: Sensitivity analysis uses gradients to determine how small changes affect system behavior.
- Physics: The gradient of potential fields indicates the direction of force.

For our function, knowing the gradient enables us to:

- Identify the direction of steepest increase or decrease.
- Locate critical points and analyze their nature.
- Develop algorithms for minimizing $f(x, y)$ if needed.

Visualizing the Gradient: Intuitive Understanding

Graphical interpretation enhances comprehension:

- Gradient vectors at various points illustrate the direction of the steepest ascent.
- Near $(0, 0)$, the vectors tend to shrink to zero, confirming the critical point.
- Along certain lines, the vectors point predominantly in the x-direction or are aligned with the $x + 6y$ term, illustrating how the combined variable influences the function's slope.

Summary: The Significance of the Gradient in Multivariable Functions

In conclusion, the gradient of $f(x, y) = x^2 + (x + 6y)^4$ is given by:

$$
\nabla f(x, y) = \left(2x + 4(x + 6y)^3, \quad 24(x + 6y)^3 \right)
$$

This vector encapsulates the function's local sensitivity to changes in x and y, guiding us toward critical points and understanding the terrain of the function landscape. Recognizing where the gradient vanishes informs us of potential minima, maxima, or saddle points—crucial knowledge for optimization and analysis across scientific disciplines.

By mastering the computation and interpretation of gradients, mathematicians, engineers, and data scientists can better navigate complex functions, optimize systems, and develop insights that drive innovation.

In essence, the gradient acts as the function's directional heartbeat, signaling where and how it changes, and empowering us to harness these insights for practical and theoretical advancements.

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