The Polynomial Of Degree 3, P(x), Has A Root Of Multiplicity 2 At X=2 And A Root Of Multiplicity 1 At

The Polynomial Of Degree 3, P(x), Has A Root Of Multiplicity 2 At X=2 And A Root Of Multiplicity 1 At x = 5. This statement provides critical information about the structure of the polynomial, which allows us to understand its factors, roots, and overall behavior. In this comprehensive guide, we will explore how to construct such a polynomial, analyze its properties, and understand the significance of roots with different multiplicities.

Understanding Roots and Multiplicities in Polynomial Functions

What Are Roots of a Polynomial?

A root (or zero) of a polynomial function P(x) is a value of x for which P(x) = 0. For example, if P(a) = 0, then x = a is a root of the polynomial. Roots are fundamental in understanding the graph and behavior of polynomial functions because they indicate where the graph intersects or touches the x-axis.

What Is Root Multiplicity?

The multiplicity of a root refers to the number of times that root appears as a factor in the polynomial. For instance:
- A root of multiplicity 1 (simple root) means the factor appears once.
- A root of multiplicity 2 means the factor appears twice.
- Higher multiplicities imply more repeated factors.

The multiplicity affects the graph's shape at the root:
- For odd multiplicities (like 1 or 3), the graph crosses the x-axis at that root.
- For even multiplicities (like 2 or 4), the graph touches the x-axis and turns around, creating a "touch and go" point.

Constructing the Polynomial P(x)

Given the roots and their multiplicities, we can write the polynomial in factored form.

Step 1: Identify the Roots and Their Multiplicities

- Root at $x = 2$ with multiplicity 2
- Root at $x = 5$ with multiplicity 1

Step 2: Write the Polynomial in Factorized Form

The polynomial $P(x)$ can be expressed as:
$$ P(x) = k \times (x - 2)^2 \times (x - 5) $$
where k is a non-zero constant that can be determined based on additional conditions (such as leading coefficient or specific point values).

Step 3: General Form of P(x)

The general form, considering the roots and their multiplicities, is:
$$ P(x) = k (x - 2)^2 (x - 5) $$
where:
- $(x - 2)^2$ accounts for the double root at $x=2$
- $(x - 5)$ accounts for the simple root at $x=5$

Analyzing the Polynomial’s Behavior

Graphical Interpretation

Understanding the roots' multiplicities helps visualize the graph:
- At $x=2$: Since the root has multiplicity 2 (even), the graph touches the x-axis and turns back, creating a "parabolic" contact.
- At $x=5$: With multiplicity 1, the graph crosses the x-axis cleanly.

Leading Coefficient and End Behavior

The value of k influences the vertical stretch and the end behavior of the polynomial:
- If $k > 0$, as x approaches $\pm \infty$, $P(x)$ approaches $+\infty$.
- If $k < 0$, the polynomial approaches $-\infty$ at both ends.

Determining the Polynomial When an Additional Point Is Given

Suppose you are provided with a specific point, such as P(0) = 10. This allows you to solve for the constant k.

Example Calculation

Given:

$$ P(0) = 10 $$

and

$$ P(x) = k (x - 2)^2 (x - 5) $$

Plugging in x=0:

$$ 10 = k (0 - 2)^2 (0 - 5) $$

$$ 10 = k (4) (-5) $$

$$ 10 = -20k $$

$$ k = -\frac{10}{20} = -\frac{1}{2} $$

Thus, the polynomial becomes:

$$ P(x) = -\frac{1}{2} (x - 2)^2 (x - 5) $$

Expanding the Polynomial

To express P(x) in standard polynomial form, expand the factors:

Step 1: Expand $ (x - 2)^2 $

$$ (x - 2)^2 = x^2 - 4x + 4 $$

Step 2: Multiply by $ (x - 5) $

$$ (x^2 - 4x + 4)(x - 5) $$

Using distributive property:

$$
x^2(x - 5) - 4x(x - 5) + 4(x - 5)
$$

$$
= x^3 - 5x^2 - 4x^2 + 20x + 4x - 20
$$

Combine like terms:

$$
x^3 - (5x^2 + 4x^2) + (20x + 4x) - 20
$$

$$
= x^3 - 9x^2 + 24x - 20
$$

Step 3: Multiply by $k = -\frac{1}{2}$

$$
P(x) = -\frac{1}{2} (x^3 - 9x^2 + 24x - 20)
$$

$$
= -\frac{1}{2} x^3 + \frac{9}{2} x^2 - 12x + 10
$$

This is the explicit form of the polynomial with the specified roots, multiplicities, and point.

Applications and Significance of Roots with Different Multiplicities

Implications in Graphing

- Roots with multiplicity 1 cause the graph to cross the x-axis.
- Roots with even multiplicity cause the graph to touch and turn around at the x-axis.

Applications in Engineering and Physics

- Polynomial models with roots of various multiplicities appear in structural analysis and signal processing.
- The behavior around roots influences stability and resonance in physical systems.

Solving Polynomial Equations

Understanding roots and multiplicities helps in solving polynomial equations efficiently, especially when factoring or using synthetic division.

Summary and Key Takeaways

- When a polynomial has a root of multiplicity 2 at x=2 and a root of multiplicity 1 at x=5, its factored form is $P(x) = k (x - 2)^2 (x - 5)$.
- The constant k can be determined based on additional conditions such as specific

function values.
- The behavior of the polynomial at its roots is governed by the multiplicity, influencing the graph's shape.
- Expanding the polynomial into standard form involves algebraic expansion and distribution.
- Such polynomials are fundamental in various fields and serve as foundational examples in algebra and calculus.

By mastering the concepts of roots and multiplicities, students and professionals can analyze and construct polynomial functions tailored to specific requirements, enhancing their problem-solving toolkit.

Keywords: polynomial degree 3, roots, multiplicity, factored form, graph behavior, polynomial expansion, algebra, calculus, root analysis, polynomial modeling

Frequently Asked Questions

What does it mean for a polynomial P(x) to have a root of multiplicity 2 at x = 2?

It means that (x - 2) appears squared as a factor in the polynomial, and the root x = 2 occurs twice in the factorization, indicating a repeated root.

Given that P(x) has a root of multiplicity 2 at x = 2 and a root of multiplicity 1 at some other point, how can we express P(x)?

P(x) can be expressed as $P(x) = a(x - 2)^2(x - r)$, where r is the other root and a is a non-zero constant.

How do the multiplicities of roots influence the shape of the polynomial's graph?

A root with multiplicity 2 causes the graph to touch and bounce off the x-axis at that root, creating a parabola-like behavior, while a root with multiplicity 1 causes the graph to cross the x-axis linearly.

If the polynomial P(x) has roots at x=2 with multiplicity 2 and at x=3 with multiplicity 1, what is the general form of P(x)?

The general form is $P(x) = a(x - 2)^2(x - 3)$, where a is a non-zero constant.

How can the leading coefficient and roots be used to fully determine a degree 3 polynomial like P(x)?

By knowing the roots and their multiplicities, along with the leading coefficient, you can write P(x) as a product of factors corresponding to the roots, scaled by the leading coefficient.

What is the significance of the root x = 2 having multiplicity 2 in terms of the polynomial's derivative at that point?

Since x = 2 is a root of multiplicity 2, the polynomial's derivative P'(x) will also be zero at x = 2, indicating a point of tangency or a repeated root.

Can the polynomial P(x) with a root of multiplicity 2 at x=2 and another at x=r have multiple forms? If so, how do we determine the specific form?

Yes, multiple forms are possible depending on the leading coefficient and the value of r. To find the specific form, additional information such as a point on the polynomial or its value at a certain x is needed.

What methods can be used to find the explicit polynomial P(x) given the roots and their multiplicities?

You can construct P(x) by multiplying the factors corresponding to each root with their multiplicities, then determine the leading coefficient using additional points or conditions, such as P(0) or P(1).

How does knowing the roots and their multiplicities help in graphing or analyzing the polynomial P(x)?

Knowing the roots and multiplicities provides critical points where the graph touches or crosses the x-axis, allowing for accurate sketching and understanding of the polynomial's behavior near those roots.

Additional Resources

The polynomial of degree 3, P(x), has a root of multiplicity 2 at x=2 and a root of multiplicity 1 at an unspecified point. This statement sets the stage for a comprehensive exploration of cubic polynomials, their roots, multiplicities, and the implications for their algebraic and geometric properties. Understanding how roots and their multiplicities shape the behavior of polynomials is fundamental in algebra, calculus, and many applied fields such as engineering, physics, and computer science. In this article, we'll delve into what it means for a polynomial of degree 3 to have roots with specified multiplicities, how to construct such polynomials, interpret their graphs, and analyze their properties.

Understanding Roots and Multiplicities in Cubic Polynomials

What Is a Root of a Polynomial?

A root (or zero) of a polynomial $P(x)$ is a value r such that $P(r) = 0$. Roots can be real or complex, simple or multiple. For a cubic polynomial, which has degree 3, there are up to 3 roots in the complex plane (counting multiplicities).

What Does Multiplicity Mean?

The multiplicity of a root refers to how many times that root appears as a factor in the polynomial's factorization. More precisely:

- If r is a root of multiplicity 1, then $(x - r)$ appears once in the factorization.
- If r is a root of multiplicity 2, then $(x - r)^2$ appears.
- If r is of multiplicity 3, then $(x - r)^3$ appears.

For cubic polynomials, roots can have multiplicities of 1, 2, or 3, but since the degree is 3, the sum of the multiplicities must be 3.

Constructing the Polynomial P(x)

Given the information:

- The polynomial $P(x)$ has a root of multiplicity 2 at $x=2$.
- The polynomial has another root of multiplicity 1 at an unspecified point, say $x = k$.

Step 1: Expressing the Polynomial in Factorized Form

The general form considering the roots and their multiplicities:

$$
P(x) = a \times (x - 2)^2 \times (x - k)
$$

where:
- a is a non-zero constant (leading coefficient).
- $(x - 2)^2$ accounts for the root at 2 with multiplicity 2.
- $(x - k)$ accounts for the simple root at $x = k$.

Step 2: Determining the Leading Coefficient

- If not specified, the leading coefficient a can be chosen for convenience, often taken as 1 for simplicity.
- For example, setting $a = 1$, the polynomial becomes:

$$
P(x) = (x - 2)^2 (x - k)
$$

which is monic (leading coefficient 1).

Step 3: Expanding the Polynomial

To understand its explicit form, expand $P(x)$:

$$
P(x) = (x - 2)^2 (x - k) = [x^2 - 4x + 4](x - k)
$$

Distribute:

$$
P(x) = x^3 - kx^2 - 4x^2 + 4kx + 4x - 4k
$$

Combine like terms:

$$
P(x) = x^3 + (-k - 4) x^2 + (4k + 4) x - 4k
$$

This explicit form allows us to analyze specific properties, such as the end behavior, critical points, and graph shape, for any chosen k.

Analyzing the Roots and Their Implications

The Root at $x=2$ with Multiplicity 2

- The fact that $(x-2)^2$ appears indicates the polynomial touches the x-axis at $x=2$ but does not cross it. The graph is tangent to the x-axis at this point.
- The multiplicity 2 causes the graph to have a parabolic touch at $x=2$. The sign of $P(x)$ on either side of $x=2$ remains the same.

The Root at $x = k$ with Multiplicity 1

- The simple root at $x=k$ means the graph crosses the x-axis at $x=k$.
- The location k influences the overall shape and position of the polynomial's graph.
- The value of k can be any real number, leading to different polynomial behaviors.

Graphical Interpretation and Properties

Behavior at Infinity

- Since the degree of $P(x)$ is 3 and the leading coefficient a is positive, the polynomial:

- Tends to $+\infty$ as $x \to +\infty$
- Tends to $-\infty$ as $x \to -\infty$

Critical Points and Turning Points

- The multiplicity of roots influences the polynomial's turning points:

- At $x=2$, due to multiplicity 2, the graph flattens at the x-axis, with a point of tangency.
- The crossing at $x=k$ is typically a simple crossing with a slope that is non-zero.

Effect of Varying k

- Changing k shifts the position of the linear root along the x-axis.
- When k is less than 2, the crossing at $x=k$ occurs to the left of the tangent point at $x=2$.
- When k is greater than 2, the crossing occurs to the right of $x=2$.

Special Cases and Examples

Example 1: $k=0$

The polynomial:

$$
P(x) = (x - 2)^2 x
$$

Expanded:

$$
P(x) = x^3 - 4x^2 + 4x
$$

- Roots at $x=0$ (simple root), and $x=2$ with multiplicity 2.
- Graph passes through the origin, touching the x-axis at $x=2$.

Example 2: $k=3$

The polynomial:

$$
P(x) = (x - 2)^2 (x - 3)
$$

Expanded:

$$
P(x) = x^3 - 7x^2 + 16x - 12
$$

- Roots at $x=2$ (multiplicity 2) and $x=3$ (simple crossing).

Applications and Further Analysis

Polynomial Interpolation and Root-Finding

Knowing the roots and their multiplicities is essential in:

- Factoring polynomials.
- Deriving polynomial equations from roots.
- Designing polynomials with desired properties.

Calculus: Critical Points and Inflection Points

- Derivatives can be used to find local maxima, minima, and points of inflection.
- The multiplicity affects the nature of the critical points:

- At $x=2$, the multiplicity 2 root results in a critical point where the slope is zero, and the graph flattens.
- At $x=k$, the crossing point generally shows a non-zero slope unless the root coincides with a multiple root.

Generalization to Other Degrees and Roots

While this discussion centers on a cubic polynomial with specific root multiplicities, similar principles apply to polynomials of higher degrees, with multiplicities summing to the degree.

Summary and Key Takeaways

- A degree 3 polynomial with a root of multiplicity 2 at $x=2$ and a root of multiplicity 1 at $x=k$ can be written as:

$$
P(x) = a \times (x - 2)^2 \times (x - k)
$$

- The polynomial's behavior near roots depends on their multiplicities:

- Double root at $x=2$: tangent contact with the x-axis.
- Single root at $x=k$: crossing the x-axis.

- The shape and position of the graph depend on the value of k and the leading coefficient \(a \).

- Expanding the polynomial gives explicit formulas suitable for analysis or plotting.

- Understanding these properties is crucial for solving polynomial equations, analyzing graphs, and applying calculus techniques.

Final Thoughts

Grasping the structure of a cubic polynomial with specified roots and multiplicities unlocks a deeper understanding of polynomial behavior. Whether you're solving algebraic equations, analyzing function graphs, or designing systems with polynomial models, recognizing how roots and their multiplicities influence the shape and properties of the polynomial is an invaluable skill. By mastering these concepts, you're equipped to interpret and manipulate cubic functions with confidence, paving the way for advanced mathematical exploration.

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