The Volume Of The Solid Obtained By Rotating The Region Bounded By Y=x^2, And Y=5x About The Line X=5

The Volume Of The Solid Obtained By Rotating The Region Bounded By Y=x^2, And Y=5x About The Line X=5 is a fascinating problem in calculus that combines the concepts of region analysis and solid volume calculation through the method of washers or disks. Understanding how to approach this problem not only enhances your grasp of integration techniques but also deepens your comprehension of geometric transformations and applications in real-world contexts.

Introduction to the Problem and Its Significance

Calculus often involves finding volumes of three-dimensional objects generated by rotating a two-dimensional region around an axis. This process is fundamental in fields such as engineering, physics, and architecture, where designing objects with specific volume constraints is common.

In this specific problem, the goal is to determine the volume of the solid formed when the region bounded by the curves $y = x^2$ and $y = 5x$ is revolved around the vertical line $x = 5$. This task involves several key steps:
- Identifying the bounded region
- Determining the points of intersection
- Choosing the appropriate method (disks, washers, shells)
- Setting up the integral expressions
- Computing the integral to find the volume

Understanding the Bounded Region

Graphs of the Curves $y = x^2$ and $y = 5x$

Before calculating the volume, visualize the region:
- The parabola $y = x^2$ opens upward with vertex at the origin.
- The line $y = 5x$ is a straight line passing through the origin with a slope of 5.

Plotting these curves reveals their intersection points, which define the limits of integration for the volume calculation.

Finding the Intersection Points

To identify where the curves intersect:
- Set $y = x^2$ equal to $y = 5x$:

$x^2 = 5x$

- Solve for x:

$x^2 - 5x = 0$

$x(x - 5) = 0$

- Solutions:

$x = 0$ and $x = 5$

- Corresponding y-values:

For $x=0$: $y=0$

For $x=5$: $y=25$

Thus, the region bounded by the curves extends from $x=0$ to $x=5$.

Choosing the Axis of Revolution and Method

Axis of Rotation: x = 5

Rotating around the line $x=5$, which is vertical, impacts how the volume is calculated. Since the axis is not at the origin, the typical disk method needs to be adapted to account for the offset.

Method Selection: Shell vs. Washer

Two common methods for volume calculations are:
- Disk/Washer Method: Suitable when rotating around axes parallel to the coordinate axes and when the region is described with respect to x or y.
- Shell Method: Often more convenient when the axis is vertical or horizontal and the region is expressed in terms of the other variable.

In this case, because the rotation is around a vertical line $x=5$, and the region is described in terms of x, the shell method is generally more straightforward.

Applying the Shell Method

Principle of the Shell Method

The shell method involves integrating cylindrical shells generated by revolving thin vertical strips around the axis:
- Each shell has a radius (distance from the shell to the axis)
- Height (determined by the functions)
- Thickness (dx)

The volume of each shell:

$$ dV = 2\pi \times (\text{radius}) \times (\text{height}) \times dx $$

Determining the Shells’ Dimensions

- Radius: The distance from the shell at x to the line x=5:

$$ R(x) = 5 - x $$

- Height: The difference between the upper and lower functions at x:

$$ h(x) = y_{top} - y_{bottom} = 5x - x^2 $$

Note: Since $y=5x$ is above $y=x^2$ between x=0 and x=5 (verified by plotting or testing points), the height is $5x - x^2$.

Formulating the Integral for Volume

The volume is obtained by integrating from x=0 to x=5:

$$
V = \int_{0}^{5} 2\pi \times R(x) \times h(x) \, dx
$$

Plugging in R(x) and h(x):

$$
V = \int_{0}^{5} 2\pi (5 - x) (5x - x^2) \, dx
$$

This simplifies to:

\[

V = 2\pi \int_{0}^{5} (5 - x)(5x - x^2) \, dx
\]

Calculating the Integral

Expanding the Integrand

Multiply out the terms:

\[
(5 - x)(5x - x^2) = 5 \times 5x - 5 \times x^2 - x \times 5x + x \times x^2
\]

Simplify each term:

\[
= 25x - 5x^2 - 5x^2 + x^3
\]

Combine like terms:

\[
= 25x - 10x^2 + x^3
\]

Thus, the volume integral becomes:

\[
V = 2\pi \int_{0}^{5} (25x - 10x^2 + x^3) \, dx
\]

Integrating Term-by-Term

Calculate each integral:

\[
\int 25x \, dx = 12.5 x^2
\]

\[
\int -10x^2 \, dx = - \frac{10}{3} x^3
\]

\[
\int x^3 \, dx = \frac{1}{4} x^4

\]

Applying limits from 0 to 5:

$$
V = 2\pi \left[12.5 x^2 - \frac{10}{3} x^3 + \frac{1}{4} x^4 \right]_0^{5}
$$

Compute each term at x=5:

- $12.5 \times 5^2 = 12.5 \times 25 = 312.5$
- $\frac{10}{3} \times 5^3 = \frac{10}{3} \times 125 = \frac{1250}{3}$
- $\frac{1}{4} \times 5^4 = \frac{1}{4} \times 625 = \frac{625}{4}$

Now, substitute into the expression:

$$
V = 2\pi \left(312.5 - \frac{1250}{3} + \frac{625}{4} \right)
$$

Express all terms with a common denominator (12):

- $312.5 = \frac{3750}{12}$
- $\frac{1250}{3} = \frac{5000}{12}$
- $\frac{625}{4} = \frac{1875}{12}$

Combine:

$$
V = 2\pi \left(\frac{3750}{12} - \frac{5000}{12} + \frac{1875}{12} \right) = 2\pi \times \frac{(3750 - 5000 + 1875)}{12}
$$

Calculate numerator:

$$
3750 - 5000 + 1875 = (3750 + 1875) - 5000 = 5625 - 5000 = 625
$$

Finally:

$$
V = 2\pi \times \frac{625}{12} = \frac{1250\pi}{12} = \frac{625\pi}{6}
$$

Therefore, the volume of the solid is:

$$
\boxed{
V = \frac{625\pi}{6}
}
$$

\]

which is approximately 327.25 cubic units.

Summary and Practical Implications

This calculation demonstrates how to approach the problem of finding the volume of a solid generated by rotating a bounded region around a vertical line other than the axes. The key takeaways include:
- Properly identifying the bounded region through intersection points.
- Choosing the shell method for rotation about vertical lines when the region is expressed in terms of x.
- Carefully setting up the integral, considering the radius and height of the shells.
- Executing the algebraic expansion and integration accurately.

Such techniques are vital in engineering design, manufacturing, and scientific research, where understanding the volume of complex objects formed by rotation is essential.

Additional Considerations and Extensions

- Alternative Methods: In some cases, the washer method may be more suitable, especially if the region is better described in terms of y or if the axis of rotation is horizontal.
- Numerical Approximations: For more complex functions where analytical integration is difficult, numerical methods like Simpson’s rule or trapezoidal rule can approximate the volume.
- 3D Modeling Software: Modern CAD tools can visualize and compute such volumes, providing practical checks for the

Frequently Asked Questions

How do you set up the integral to find the volume of the solid obtained by rotating the region bounded by y = x^2 and y = 5x about the line x = 5?

First, identify the points of intersection of y = x^2 and y = 5x to determine the limits of integration. Then, using the washer method, set up the integral with respect to y or x, considering the distance from the axis of rotation x=5. Typically, expressing x in terms of y and integrating with respect to y simplifies the calculation, with the outer radius as (5 - x) and the inner radius as the x-value of the region boundary.

What is the significance of the line x=5 in the problem of rotating the region bounded by y = x^2 and y = 5x?

The line x=5 serves as the axis of rotation. Rotating the bounded region about this vertical line generates a three-dimensional solid. The position of the line relative to the region determines the shape of the resulting solid and influences the setup of the integral for volume calculation.

How do you find the points of intersection between y = x^2 and y = 5x?

Set the two equations equal: x^2 = 5x. Rearranged as x^2 - 5x = 0, which factors to x(x - 5) = 0. Therefore, the intersection points occur at x = 0 and x = 5.

Which method is more appropriate for calculating the volume in this problem: disks, washers, or shells?

The washer method is most appropriate here because the solid is generated by rotating a region bounded by two curves around a vertical line. Using washers accounts for the hole in the middle, especially when the axis of rotation is not the y-axis or when the region does not start from the axis.

What are the steps to evaluate the volume of the solid obtained by rotating the region about x=5?

First, find the intersection points of the curves to determine limits. Express x in terms of y or vice versa for integration. Set up the integral for volume using the washer method, where the outer radius is the distance from x=5 to the outer boundary, and the inner radius to the inner boundary. Then, evaluate the integral to find the volume.

Can you describe the shape of the resulting solid when rotating the region bounded by y = x^2 and y = 5x about x=5?

The rotation creates a hollow, torus-like solid with a central hole, since the region is rotated around a vertical line outside the region itself. The shape resembles a doughnut with varying cross-sectional radii determined by the curves and the position of the axis.

Additional Resources

Comprehensive Analysis of the Volume of the Solid Formed by Rotating the

Region Bounded by $y = x^2$ and $y = 5x$ About the Line $x = 5$

Introduction: Exploring Solid of Revolution and Its Significance

The study of volumes generated by revolving a region around a specified axis is a fundamental topic in integral calculus, with applications spanning engineering, physics, and mathematical modeling. This specific problem involves calculating the volume of a solid obtained by rotating the bounded region between two curves, $y = x^2$ and $y = 5x$, about the vertical line $x = 5$. The process exemplifies the use of the disk/washer method and highlights the importance of understanding the geometry of the region, the setup of integrals, and the execution of the calculations.

Understanding the Bounded Region

Before delving into the volume calculation, a comprehensive understanding of the region involved is essential.

1. Equations of the Curves

- Parabola: $y = x^2$
- Line: $y = 5x$

These two curves intersect at points where $x^2 = 5x$, which simplifies to:

$$
x^2 - 5x = 0
$$
$$
x(x - 5) = 0
$$

Thus, the points of intersection occur at:

$$
x = 0 \quad \text{and} \quad x = 5
$$

Corresponding y-values are:

- At $x = 0$: $y = 0$
- At $x = 5$: $y = 25$

The bounded region is therefore between $x=0$ and $x=5$, enclosed between these two curves.

2. Visualizing the Region

The region can be visualized as a "curvilinear strip" between the parabola $y = x^2$ and the line $y = 5x$ from $x=0$ to $x=5$. Since $y = 5x$ is steeper and above $y = x^2$ in the interval, the region is located where:

$$
x^2 \leq y \leq 5x
$$

This setup will guide the choice of the method for calculating the volume.

Choosing the Axis of Rotation: The Line $x=5$

The rotation is about the vertical line $x=5$, which is to the right of the region. This choice influences the method of integration:

- Method: Shell Method (cylindrical shells)
- Rationale: Since the axis of rotation is vertical and to the right of the region, using shells is typically more straightforward than disks or washers, which would require more complex setup.

Methodology: Shell Method for Volume Calculation

The shell method involves integrating cylindrical shells' lateral surface areas along the axis of the region, with each shell corresponding to a vertical strip in the original region.

1. Shell Method Formula

The volume V is given by:

$$
V = 2\pi \int_{a}^{b} \text{(radius)} \times \text{(height)} \, dx
$$

Where:

- Radius: distance from the shell to the axis $x=5$, which is $|5 - x|$
- Height: difference in y-values at a fixed x, i.e., $y_{\text{top}} - y_{\text{bottom}}$

In this case:

$$
V = 2\pi \int_{x=0}^{x=5} (5 - x) \times [(5x) - (x^2)] \, dx
$$

Note: Because x ranges from 0 to 5, and the axis is at 5, the radius simplifies to $5 - x$.

2. Setting Up the Integral

The integrand involves:

- Radius: $5 - x$
- Height of shell: $(5x) - x^2$

Thus,

$$
V = 2\pi \int_0^5 (5 - x) \times (5x - x^2) \, dx
$$

Calculating the Integral: Step-by-Step Breakdown

The core of this problem is calculating:

$$

$$
V = 2\pi \int_0^5 (5 - x)(5x - x^2) \, dx
$$

---

1. Expanding the Integrand

Multiply out the terms:

$$
(5 - x)(5x - x^2) = 5 \times 5x - 5 \times x^2 - x \times 5x + x \times x^2
$$

Simplify:

$$
= 25x - 5x^2 - 5x^2 + x^3
$$

$$
= 25x - 10x^2 + x^3
$$

So, the volume integral becomes:

$$
V = 2\pi \int_0^5 (25x - 10x^2 + x^3) \, dx
$$

---

2. Integrating Term-by-Term

Calculate each integral:

- $\int 25x \, dx = 12.5 x^2$
- $\int -10x^2 \, dx = - \frac{10}{3} x^3$
- $\int x^3 \, dx = \frac{1}{4} x^4$

Applying limits from 0 to 5:

$$
V = 2\pi \left[12.5 x^2 - \frac{10}{3} x^3 + \frac{1}{4} x^4 \right]_0^5
$$

---

3. Evaluating at $x=5$ and $x=0$

At $x=5$:

$$
12.5 \times 25 - \frac{10}{3} \times 125 + \frac{1}{4} \times 625
$$

Calculate each term:

- $12.5 \times 25 = 312.5$
- $\frac{10}{3} \times 125 = \frac{1250}{3} \approx 416.66$
- $\frac{1}{4} \times 625 = 156.25$

Putting it together:

$$
312.5 - 416.66 + 156.25 = (312.5 + 156.25) - 416.66 = 468.75 - 416.66 \approx 52.09
$$

At $x=0$, all terms are zero, so:

$$
V = 2\pi \times 52.09 \approx 2\pi \times 52.09
$$

Final Volume Calculation and Interpretation

Multiplying:

$$
V \approx 2 \times 3.1416 \times 52.09 \approx 6.2832 \times 52.09 \approx 327.55
$$

Therefore, the approximate volume of the solid formed by rotating the bounded region about $x=5$ is around 327.55 cubic units.

Additional Insights and Considerations

1. Exact Expression

For a precise answer, the volume is:

$$
V = 2\pi \left(12.5 \times 25 - \frac{10}{3} \times 125 + \frac{1}{4} \times 625 \right)
$$

Simplify step-by-step:

$$
V = 2\pi \left(312.5 - \frac{1250}{3} + 156.25 \right)
$$

Express all terms with common denominators:

$$
V = 2\pi \left(\frac{937.5}{3} - \frac{1250}{3} + \frac{468.75}{3} \right)
$$

Combine numerator:

$$
V = 2\pi \times \frac{937.5 - 1250 + 468.75}{3} = 2\pi \times \frac{156.25}{3}
$$

Thus,

$$
V = 2\pi \times \frac{156.25}{3} = \frac{2 \times 156.25 \pi}{3} = \frac{312.5 \pi}{3}
$$

And the exact volume:

$$
V = \frac{312.5 \pi}{3} \text{ cubic units}
$$

---

2. Significance of the Result

This volume represents a three-dimensional object with rotational symmetry around $x=5$. The calculation demonstrates a typical application of the shell method, especially when

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