1. Consider The Differential Equation = Ky (10 - Y). Let Y = F(x) Be The Dx Particular Solution To The

1. Consider The Differential Equation = Ky (10 - Y). Let Y = F(x) Be The Dx Particular Solution To The

Understanding differential equations is fundamental in solving numerous real-world problems in fields such as physics, engineering, biology, and economics. The differential equation given as $\frac{dY}{dx} = Ky(10 - Y)$ is a classic example of a first-order nonlinear ordinary differential equation, often encountered in modeling population growth with a carrying capacity or logistic growth phenomena. In this article, we will explore the process of finding the particular solution $Y = F(x)$ to this differential equation, analyze its behavior, and understand its implications comprehensively.

Understanding the Differential Equation $\frac{dY}{dx} = Ky(10 - Y)$

Before delving into solving the differential equation, it’s essential to comprehend its structure and the context in which it appears.

Nature of the Equation

- Type: First-order nonlinear ordinary differential equation
- Form: Logistic growth model
- Variables:
- $Y = F(x)$: the dependent variable, often representing population, concentration, or other quantities
- x: independent variable, typically time or space
- Parameters:
- K: a positive constant representing the growth rate
- The term $(10 - Y)$: indicates a limiting factor or carrying capacity

Real-World Applications

- Population Dynamics: Modeling populations that grow rapidly initially but slow down as they approach a maximum sustainable population (carrying capacity)
- Resource Consumption: Tracking the accumulation of a resource within a system with a maximum limit

- Epidemiology: Spread of diseases where the susceptible population decreases over time

---

Solving the Differential Equation $\frac{dY}{dx} = Ky(10 - Y)$

The differential equation is separable, allowing for a straightforward integration process.

Step 1: Separate Variables

Rewrite the equation to isolate Y and x:

$$
\frac{dY}{Y(10 - Y)} = K\, dx
$$

This separation involves expressing the differential as a ratio of functions of Y and x.

Step 2: Partial Fraction Decomposition

To integrate the left side, decompose the fraction:

$$
\frac{1}{Y(10 - Y)} = \frac{A}{Y} + \frac{B}{10 - Y}
$$

Multiply both sides by $Y(10 - Y)$:

$$
1 = A(10 - Y) + BY
$$

Set $Y = 0$:

$$
1 = A \times 10 \Rightarrow A = \frac{1}{10}
$$

Set $Y = 10$:

$$
1 = B \times 10 \Rightarrow B = \frac{1}{10}

\]

Therefore:

\[
\frac{1}{Y(10 - Y)} = \frac{1}{10} \left(\frac{1}{Y} + \frac{1}{10 - Y} \right)
\]

Step 3: Integrate Both Sides

Integrate:

\[
\int \frac{1}{Y(10 - Y)} dY = \frac{1}{10} \int \left(\frac{1}{Y} + \frac{1}{10 - Y} \right) dY
\]

This yields:

\[
\frac{1}{10} \left(\ln|Y| - \ln|10 - Y| \right) + C
\]

The integral of the right side is straightforward:

\[
\int K\, dx = Kx + C'
\]

Putting it together:

\[
\frac{1}{10} \left(\ln|Y| - \ln|10 - Y| \right) = Kx + C
\]

Deriving the General Solution

Step 4: Simplify the Expression

Multiply through by 10:

\[
\ln|Y| - \ln|10 - Y| = 10Kx + C''

\]

Express as a single logarithm:

\[
\ln \left| \frac{Y}{10 - Y} \right| = 10Kx + C''
\]

Let \(C'' = \ln C_1 \) (since the exponential of a constant is just another constant):

\[
\left| \frac{Y}{10 - Y} \right| = C_1 e^{10Kx}
\]

Assuming \(Y \) is within a range where \(\frac{Y}{10 - Y} \) is positive, we can write:

\[
\frac{Y}{10 - Y} = C e^{10Kx}
\]

where \(C \) is an arbitrary constant determined by initial conditions.

Step 5: Solve for \(Y \)

Rearranged:

\[
Y = (10 - Y) C e^{10Kx}
\]

\[
Y + Y C e^{10Kx} = 10 C e^{10Kx}
\]

\[
Y (1 + C e^{10Kx}) = 10 C e^{10Kx}
\]

\[
Y = \frac{10 C e^{10Kx}}{1 + C e^{10Kx}}
\]

This is the general solution for the differential equation.

Particular Solution Based on Initial Conditions

To find the particular solution $Y = F(x)$, initial conditions such as $Y(x_0) = Y_0$ are necessary.

Example:

Suppose at $x = 0$, $Y = Y_0$. Then:

$$
Y_0 = \frac{10 C}{1 + C}
$$

Solve for C:

$$
Y_0 (1 + C) = 10 C
$$

$$
Y_0 + Y_0 C = 10 C
$$

$$
Y_0 = C (10 - Y_0)
$$

$$
C = \frac{Y_0}{10 - Y_0}
$$

Thus, the particular solution becomes:

$$
Y(x) = \frac{10 \times \frac{Y_0}{10 - Y_0} e^{10Kx}}{1 + \frac{Y_0}{10 - Y_0} e^{10Kx}} = \frac{10 Y_0 e^{10Kx}}{(10 - Y_0) + Y_0 e^{10Kx}}
$$

Behavior and Analysis of the Solution

Understanding the solution's behavior over x is crucial for applications.

Asymptotic Behavior

- As $x \to -\infty$:

$$
e^{10Kx} \to 0 \Rightarrow Y \to 0
$$

- As $x \to +\infty$:

$$
e^{10Kx} \to \infty \Rightarrow Y \to 10
$$

This indicates the solution approaches the carrying capacity $Y = 10$ over time.

Graphical Representation

- Initial Population: Determined by initial condition Y_0
- Growth Curve: S-shaped (sigmoidal), typical of logistic growth
- Equilibrium Points: $Y = 0$ (unstable), $Y = 10$ (stable)

Implications in Real-world Scenarios

- Population Dynamics: The model predicts initial exponential growth slowing down as the population approaches the maximum sustainable level
- Resource Management: Helps in predicting when resources will stabilize
- Epidemiology: Can model disease spread where the number of infected stabilizes at a certain level

---

SEO Keywords and Phrases

- Differential equations logistic growth
- Solving first-order nonlinear ODEs
- Particular solutions to differential equations
- Logistic growth differential equation
- Population modeling with differential equations
- Separable differential equations
- Initial value problem solutions
- Asymptotic behavior of solutions
- Carrying capacity in differential models
- Logistic equation in biology and economics

Conclusion

The differential equation $\frac{dY}{dx} = Ky(10 - Y)$ models systems exhibiting logistic growth, characterized by rapid initial increase that tapers off as it approaches a maximum limit. By employing separation of variables and partial fractions, we derive the general solution:

$$
Y(x) = \frac{10 Y_0 e^{10Kx}}{(10 - Y_0) + Y_0 e^{10Kx}}
$$

This solution captures the dynamic behavior of the modeled system, illustrating how initial conditions influence the trajectory over time. Understanding the properties of this logistic model is vital for applications across various disciplines, providing insights into growth patterns, resource limitations, and population stabilization. Properly analyzing and applying these solutions enables better decision-making in fields that depend on predictive modeling of complex systems

Frequently Asked Questions

What type of differential equation is given by dY/dx = KY(10 - Y)?

It is a first-order nonlinear differential equation of the logistic type.

How do you find the particular solution Y = F(x) for the differential equation dY/dx = KY(10 - Y)?

You can separate variables and integrate:

$\int dY / [Y(10 - Y)] = \int K\, dx$,

then solve for Y in terms of x.

What is the general solution to the differential equation dY/dx = KY(10 - Y)?

The general solution is

$Y(x) = 10 / (1 + C e^{-10Kx})$,

where C is an arbitrary constant determined by initial conditions.

What are the equilibrium solutions of the differential equation dY/dx = KY(10 - Y)?

The equilibrium solutions occur when dY/dx = 0, which gives Y = 0 and Y = 10.

How does the sign of K influence the behavior of the solution to dY/dx = KY(10 - Y)?

If K > 0, the solution tends toward Y=10 over time; if K < 0, it tends toward Y=0, indicating growth or decay respectively.

What initial condition is needed to find a specific particular solution Y = F(x)?

An initial value Y(x_0) = Y_0 is required to determine the constant C in the solution.

What biological or real-world phenomena can be modeled by the differential equation dY/dx = KY(10 - Y)?

It models logistic growth, such as population dynamics where growth slows as the population approaches a carrying capacity of 10.

Additional Resources

1. Consider The Differential Equation = Ky (10 - Y). Let Y = F(x) Be The Dx Particular Solution To The

In the realm of differential equations, understanding how to find particular solutions is essential for modeling real-world phenomena. Suppose we are presented with the differential equation:

\[\frac{dY}{dx} = K y (10 - y) \]

where \(Y = F(x)\) represents an unknown function of \(x\), and \(K\) is a constant parameter. The question arises: how do we determine a specific solution \(Y = F(x)\) that satisfies this equation? This form is a classic example of a logistic differential equation, often used in biology to model population growth, but also applicable in fields like chemistry, economics, and physics.

In this article, we will explore the process of solving this differential equation step-by-step. We will analyze the structure, identify the type of differential equation, and derive the particular solution, emphasizing both the mathematical rigor and the intuitive understanding needed to grasp these concepts clearly.

Understanding the Differential Equation: The Logistic Model

The differential equation:

$$
\frac{dY}{dx} = K y (10 - y)
$$

is a first-order, nonlinear ordinary differential equation. To appreciate its significance, it's important to recognize its structure and how it models growth processes constrained by a carrying capacity.

Logistic Growth and Its Significance

- Logistic Equation: The form resembles the logistic growth model, which describes populations that grow rapidly when small but slow down as they approach a maximum limit or carrying capacity.

- Parameters:
- K: Growth rate constant.
- 10: The carrying capacity, representing the maximum sustainable value of Y.
- Y: The variable of interest, perhaps representing population size, concentration, or other quantities.

Implication of the Equation

- When Y is small, close to zero, the term $10 - Y$ is near 10, so the growth rate is approximately proportional to Y, leading to exponential growth.
- As Y approaches 10, the term $10 - Y$ diminishes, slowing growth.
- When $Y = 10$, the growth rate drops to zero, indicating an equilibrium or steady state.

Transforming the Equation: Separating Variables

To find a particular solution $Y = F(x)$, the most straightforward approach is separation of variables. This method involves rearranging the differential equation so that all functions of Y are on one side and all functions of x on the other.

Starting with:

$$
\frac{dY}{dx} = K Y (10 - Y)
$$

we rewrite as:

$$
\frac{1}{Y (10 - Y)} dY = K dx
$$

This separation allows us to integrate both sides independently, with respect to their variables.

Partial Fraction Decomposition

The integral on the left involves:

$$
\int \frac{1}{Y (10 - Y)} dY
$$

which isn’t straightforward in its current form. To evaluate it, we use partial fraction decomposition.

Express:

$$
\frac{1}{Y (10 - Y)} = \frac{A}{Y} + \frac{B}{10 - Y}
$$

Multiply through by $Y (10 - Y)$:

$$
1 = A (10 - Y) + B Y
$$

To find A and B:

- Set $Y = 0$:

$$
1 = A \times 10 + B \times 0 \Rightarrow A = \frac{1}{10}
$$

- Set $Y = 10$:

$$
1 = A \times 0 + B \times 10 \Rightarrow B = \frac{1}{10}
$$

Thus,

$$
\frac{1}{Y (10 - Y)} = \frac{1/10}{Y} + \frac{1/10}{10 - Y} = \frac{1}{10} \left(\frac{1}{Y} + \frac{1}{10 - Y} \right)
$$

Integrating Both Sides

Now, the integral becomes:

\[
\int \frac{1}{Y (10 - Y)} dY = \frac{1}{10} \int \left(\frac{1}{Y} + \frac{1}{10 - Y}
\right) dY
\]

which simplifies to:

\[
\frac{1}{10} \left(\int \frac{1}{Y} dY + \int \frac{1}{10 - Y} dY \right)
\]

Calculating each integral:

- \(\int \frac{1}{Y} dY = \ln|Y| \)
- \(\int \frac{1}{10 - Y} dY = - \ln|10 - Y| \) (since derivative of \(10 - Y\) is \(-1\))

Putting it all together:

\[
\frac{1}{10} \left(\ln|Y| - \ln|10 - Y| \right) + C
\]

where \(C\) is the constant of integration.

Deriving the General Solution

Recall the separated form:

\[
\int \frac{1}{Y (10 - Y)} dY = K \int dx
\]

which becomes:

\[
\frac{1}{10} \left(\ln|Y| - \ln|10 - Y| \right) = K x + C
\]

Multiplying both sides by 10:

\[
\ln|Y| - \ln|10 - Y| = 10 K x + C'
\]

where \(C' = 10 C\) is a new constant of integration.

Use properties of logarithms:

\[

$$
\ln \left| \frac{Y}{10 - Y} \right| = 10 K x + C'
$$

Exponentiating both sides:

$$
\left| \frac{Y}{10 - Y} \right| = e^{10 K x + C'} = A e^{10 K x}
$$

where $A = e^{C'}$ is a positive constant determined by initial conditions.

Expressing the Particular Solution

Removing the absolute value (by considering initial conditions or the domain of Y):

$$
\frac{Y}{10 - Y} = A e^{10 K x}
$$

Solving for Y:

$$
Y = (10 - Y) A e^{10 K x}
$$

$$
Y + Y A e^{10 K x} = 10 A e^{10 K x}
$$

$$
Y (1 + A e^{10 K x}) = 10 A e^{10 K x}
$$

$$
Y = \frac{10 A e^{10 K x}}{1 + A e^{10 K x}}
$$

This is the general solution to the differential equation, with A determined by initial conditions.

Interpreting the Solution: Behavior and Applications

The solution:

$$
Y(x) = \frac{10 A e^{10 K x}}{1 + A e^{10 K x}}
$$

captures the dynamics of growth constrained by a capacity of 10 units. Its properties include:

- Growth trend: When $x \to -\infty$, $e^{10 K x} \to 0$, so $Y \to 0$.
- Saturation: As $x \to \infty$, $e^{10 K x} \to \infty$, so $Y \to 10$.
- Initial condition: If at $x = 0$, $Y(0) = Y_0$, then:

$$
Y_0 = \frac{10 A}{1 + A}
\Rightarrow A = \frac{Y_0}{10 - Y_0}
$$

This allows us to tailor the solution to specific initial data.

Practical Implications and Broader Context

This solution provides a powerful tool in modeling systems with growth limitations. For example:

- Population dynamics: Modeling species that grow rapidly but are limited by resources.
- Epidemiology: Tracking disease spread with herd immunity effects.
- Chemistry: Reaction rates approaching equilibrium concentrations.

In all these cases, understanding the particular solution helps predict long-term behavior, identify equilibrium points, and evaluate the impact of parameters like K.

Summary: From Equation to Insight

The journey from the original differential equation:

$$
\frac{dY}{dx} = K y (10 - y)
$$

to its particular solution reveals several key steps:

1. Recognizing the logistic form and its significance.
2. Separating variables to facilitate integration.
3. Applying partial fractions to handle the integrand.
4. Integrating to find a relationship between Y and x.
5. Solving algebraically for Y.
6. Interpreting the solution in context and applying initial conditions.

This process exemplifies the blend of mathematical techniques and conceptual understanding necessary to analyze nonlinear differential equations—tools that are indispensable in scientific research and engineering.

Final Thoughts

The logistic differential equation exemplifies how mathematical models can mirror complex real-world systems. The

1 Consider The Differential Equation Ky 10 Y Let Y F X Be The Dx Particular Solution To The

1 Consider The Differential Equation Ky 10 Y Let Y F X Be The Dx Particular Solution To The

Related Articles

- What Conclusion Can Be Drawn From The Data In The Histogram? Histograms | CK-12 Foundation Question 5
- 100 POINTS NEED HELP ASSAPWhat Is The Advantage Of Class 3 Lever? (1 Point)O It Decreases The Distance
- 16. The PVT Data As Shown Below Were Obtained On A Crude Oil Sample Taken From The Nameless Field. The

Back to Home