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1. Let X Poisson() And Y ~ Poisson(u) Be Independent. (a) Find The Distribution Of X +Y. (b) Determine

Understanding probability distributions is fundamental in statistics, especially when dealing with discrete random variables like the Poisson distribution. In this article, we explore the scenario where two independent Poisson random variables, X and Y, are combined, and we aim to determine the distribution of their sum, X + Y. We will also delve into related properties and implications of this combination, providing a comprehensive guide suitable for students, researchers, and professionals interested in probability theory.

Section 1: Introduction to Poisson Distributions

What is a Poisson Distribution?

- The Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space.
- It is characterized by the parameter λ (lambda), which represents the average rate of occurrence within the interval.
- The probability mass function (PMF) of a Poisson-distributed random variable X is given by:

$$
P(X = k) = \frac{e^{-\lambda} \lambda^{k}}{k!}
$$

where k = 0, 1, 2, ...

Properties of Poisson Distribution

- The mean and variance of a Poisson random variable are both equal to λ.
- It models rare events effectively when the average rate λ is small or moderate.
- The distribution is inherently discrete, making it suitable for count data.

Section 2: Defining the Random Variables

Variables Involved

- X: a Poisson random variable with an unspecified parameter, denoted as Poisson().
- Y: a Poisson random variable with parameter u, denoted as Poisson(u).
- Independence: X and Y are assumed to be independent, meaning the occurrence of one does not influence the other.

Assumptions and Clarifications

- The notation Poisson() for X suggests that its parameter is not specified; it could be any positive real number, say λ_x.
- For the purpose of analysis, we will consider that X ~ Poisson(λ_x).
- The independence assumption is critical for the derivation of the sum distribution.

Section 3: Finding the Distribution of X + Y

Theoretical Background

- The sum of two independent Poisson random variables is also Poisson-distributed.
- The key property is:

$$
\text{If } X \sim \text{Poisson}(\lambda_x) \text{ and } Y \sim \text{Poisson}(\lambda_y), \text{ and } X \perp Y,
$$

then:

$$
X + Y \sim \text{Poisson}(\lambda_x + \lambda_y).
$$

Step-by-Step Derivation

1. Identify the distributions and parameters:
- X ~ Poisson(λ_x)
- Y ~ Poisson(u)

2. Express the probability of the sum:
- For any non-negative integer k,

$$
P(X + Y = k) = \sum_{i=0}^{k} P(X = i, Y = k - i)
$$

3. Use independence to factor joint probabilities:

- Since X and Y are independent,

$$
P(X = i, Y = k - i) = P(X = i) \times P(Y = k - i)
$$

4. Substitute the PMFs:

$$
P(X = i) = \frac{e^{-\lambda_x} \lambda_x^{i}}{i!}
$$

$$
P(Y = k - i) = \frac{e^{-u} u^{k - i}}{(k - i)!}
$$

5. Calculate the sum:

$$
P(X + Y = k) = \sum_{i=0}^{k} \frac{e^{-\lambda_x} \lambda_x^{i}}{i!} \times \frac{e^{-u} u^{k - i}}{(k - i)!}
$$

6. Factor out common terms:

$$
P(X + Y = k) = e^{-(\lambda_x + u)} \sum_{i=0}^{k} \frac{\lambda_x^{i}}{i!} \times \frac{u^{k - i}}{(k - i)!}
$$

7. Recognize the convolution sum:

$$
P(X + Y = k) = e^{-(\lambda_x + u)} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda_x^{i} u^{k - i}
$$

where $\binom{k}{i} = \frac{k!}{i!(k - i)!}$.

8. Simplify using binomial theorem:

$$
P(X + Y = k) = e^{-(\lambda_x + u)} \frac{(\lambda_x + u)^k}{k!}
$$

This confirms that:

$$
X + Y \sim \text{Poisson}(\lambda_x + u)
$$

Summary:

- When X and Y are independent Poisson variables with parameters λ_x and u respectively, their sum is a Poisson variable with parameter $\lambda_x + u$.

Section 4: Implications and Applications

Why is this property important?

- It simplifies the analysis of sum counts in various practical scenarios.
- It allows for easier modeling of combined events, such as total arrivals in a system.
- It supports the aggregation of independent Poisson processes, which is common in fields like telecommunications, traffic engineering, and biological sciences.

Real-World Examples

- Customer arrivals: Suppose two separate stores receive customers according to independent Poisson processes with rates λ_1 and λ_2. The total customer arrivals across both stores follow a Poisson distribution with rate $\lambda_1 + \lambda_2$.
- Network traffic: Data packets arriving from two independent sources can be modeled as Poisson processes, and their combined traffic will also follow a Poisson distribution with the sum of individual rates.
- Biological counts: Counts of two independent species or events occurring in a habitat or over time can be combined, maintaining the Poisson distribution with an additive parameter.

Section 5: Additional Considerations and Related Topics

Variance of the Sum

- Since the variance of a Poisson(λ) variable is λ, the variance of the sum X + Y is:

$$
\text{Var}(X + Y) = \lambda_x + u
$$

- This is consistent with the properties of the Poisson distribution.

Extensions and Variations

- Non-independent variables: If X and Y are not independent, the distribution of X + Y might differ, requiring more complex models.

- Poisson mixture models: Combining Poisson distributions with other distributions can model overdispersion or heterogeneity in data.

Related Distributions

- Compound Poisson distributions: When the number of events follows a Poisson process, but each event contributes a random amount, leading to more complex models.
- Poisson-Gamma models: Used in Bayesian inference for hierarchical modeling.

Section 6: Summary and Key Takeaways

- The sum of two independent Poisson random variables is also Poisson-distributed.
- The resulting distribution's parameter is the sum of the individual parameters.
- This property simplifies modeling, analysis, and prediction in various fields involving count data.
- Understanding the convolution of Poisson distributions is essential for applied statistics, queuing theory, and stochastic processes.

Conclusion

In conclusion, when dealing with independent Poisson random variables, understanding the distribution of their sum is straightforward thanks to the additive property of Poisson parameters. This fundamental result not only provides theoretical elegance but also practical utility in modeling real-world phenomena where counts of independent events are aggregated. Whether in traffic analysis, telecommunications, biology, or operations research, recognizing that the sum follows a Poisson distribution with a parameter equal to the sum of individual parameters enables accurate and efficient probabilistic modeling.

Keywords: Poisson distribution, sum of Poisson variables, independent random variables, probability, discrete distribution, stochastic processes, modeling counts

Frequently Asked Questions

What is the distribution of the sum X + Y when X ~ Poisson(λ) and Y ~ Poisson(μ) are independent?

The sum $X + Y$ follows a Poisson distribution with parameter $\lambda + \mu$, i.e., $X + Y \sim \text{Poisson}(\lambda + \mu)$.

How do you derive the distribution of the sum of two

independent Poisson random variables?

Since the sum of independent Poisson variables is also Poisson, the distribution is obtained by adding their parameters: $X + Y \sim \text{Poisson}(\lambda + \mu)$.

What is the probability mass function (pmf) of the sum X + Y where X and Y are independent Poisson variables?

The pmf is $P(X + Y = k) = e^{-(\lambda+\mu)} (\lambda+\mu)^k / k!$, for $k = 0, 1, 2, \ldots$

In the context of independent Poisson variables, what does the distribution of X + Y tell us about combined event counts?

It indicates that the total count from both sources is Poisson-distributed with a parameter equal to the sum of individual parameters, reflecting additive properties of Poisson processes.

How can the distribution of X + Y be used in real-world applications?

It helps model combined counts in scenarios like total arrivals at multiple queues, total defect counts from different production lines, or combined events in independent processes.

What is the significance of independence in determining the distribution of the sum of two Poisson variables?

Independence ensures that the joint distribution factors into the product of individual distributions, allowing the sum to be simply Poisson with summed parameters.

Can the sum of two Poisson variables be non-Poisson? Why or why not?

No, if the variables are independent Poisson, their sum is always Poisson; dependence or other distributions would alter this result.

How does the parameter u relate to the distribution of Y in the given problem?

$Y \sim \text{Poisson}(u)$, so u is the mean number of events for Y, and the total sum's mean is $\lambda + u$.

What additional information is needed to fully determine the distribution of X + Y in part (b)?

Typically, the question might require calculating moments, probabilities for specific values, or properties related to the sum, which can be derived once the parameters are known. Specific details depend on the complete part (b) prompt.

Additional Resources

Poisson Distribution is a fundamental concept in probability theory and statistics, especially relevant in modeling count data and rare events. It is used extensively across various fields such as telecommunications, finance, biology, and physics to describe the number of events occurring within a fixed interval of time or space. Understanding how the Poisson distribution behaves when combined with other independent Poisson-distributed variables is crucial for both theoretical insights and practical applications. This article delves into the distribution of the sum of two independent Poisson random variables, specifically $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(u)$, and explores related properties and implications.

Understanding the Setup: Independent Poisson Variables

Suppose we have two independent random variables:

- $X \sim \text{Poisson}(\lambda)$
- $Y \sim \text{Poisson}(u)$

where $\lambda > 0$ and $u > 0$ are the parameters (means) of the respective distributions. The independence assumption implies that the occurrence of events counted by X does not influence those counted by Y, and vice versa.

The Poisson distribution for a discrete random variable Z is given by:

$$
P(Z = k) = \frac{\mu^k e^{-\mu}}{k!}, \quad k = 0, 1, 2, \dots
$$

where μ is the expected number of events.

Part (a): Distribution of $X + Y$

The key question is: What is the distribution of the sum $Z = X + Y$?

Theoretical Foundation: Summation Property of Poisson Variables

One of the most elegant properties of the Poisson distribution is its closure under addition when variables are independent. Specifically, the sum of two independent Poisson random variables is also Poisson-distributed, with a parameter equal to the sum of their individual parameters.

Mathematically,

$$
X + Y \sim \text{Poisson}(\lambda + u)
$$

Derivation:

- Since X and Y are independent,

$$
P(X + Y = k) = \sum_{i=0}^{k} P(X = i) P(Y = k - i)
$$

- Substituting the Poisson probabilities,

$$
P(X + Y = k) = \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!} \times \frac{u^{k - i} e^{-u}}{(k - i)!}
$$

- Simplify the sum:

$$
P(X + Y = k) = e^{-(\lambda + u)} \sum_{i=0}^{k} \frac{\lambda^i}{i!} \times \frac{u^{k - i}}{(k - i)!}
$$

- Recognize the binomial expansion:

$$
P(X + Y = k) = e^{-(\lambda + u)} \frac{1}{k!} \sum_{i=0}^k \binom{k}{i} \lambda^i u^{k - i}
$$

- Using the binomial theorem:

$$
\sum_{i=0}^k \binom{k}{i} \lambda^i u^{k - i} = (\lambda + u)^k
$$

- Final expression:

$$
P(X + Y = k) = \frac{(\lambda + u)^k e^{-(\lambda + u)}}{k!}
$$

Thus, the distribution of $X + Y$ is Poisson with parameter $\lambda + u$.

Implications and Features

- Closure Property: The Poisson distribution is closed under the convolution of independent variables.
- Additivity of Means: The mean of the sum is simply the sum of the means: $E[X + Y] = \lambda + u$.
- Simplified Calculations: This property simplifies the analysis of counts over combined intervals or aggregated counts from multiple sources.

Part (b): Determining Other Properties and Applications

Beyond finding the distribution, it is often necessary to explore other characteristics of the sum and its implications in real-world scenarios.

Conditional Distributions

Given $X + Y = k$, what is the distribution of X?

Since X and Y are independent, and their sum is known, the conditional distribution of X can be derived from the multinomial property of the Poisson process:

$$
X \mid (X + Y = k) \sim \text{Binomial}\left(k, \frac{\lambda}{\lambda + u}\right)
$$

Explanation:

- The probability that a particular event belongs to X (versus Y) is proportional to their respective parameters.
- Intuitively, given the total count k, each individual count is "allocated" to X with probability $\frac{\lambda}{\lambda + u}$, and to Y with probability $\frac{u}{\lambda + u}$.

Mathematically:

$$
P(X = i \mid X + Y = k) = \binom{k}{i} \left(\frac{\lambda}{\lambda + u}\right)^i \left(\frac{u}{\lambda + u}\right)^{k - i}
$$

Features:

- This conditional distribution is binomial, which is useful in Bayesian updating and in understanding the distribution of counts among different sources.

Applications of the Sum Distribution

The result that the sum of independent Poisson variables is also Poisson with a summed parameter has several practical applications:

- Modeling Total Counts: In network traffic, the total number of packets arriving from multiple independent sources can be modeled as a Poisson process with a combined rate.
- Risk Management: Aggregating rare event counts from different risk categories to estimate total risk.
- Biology and Epidemiology: Summing counts from different regions or populations.

Pros and Cons of the Poisson Sum Property

Pros:
- Simplifies complex models involving multiple independent sources.
- Facilitates analytical derivations and simulations.
- Supports intuitive understanding of combined event processes.

Cons:
- Assumes independence, which may not always hold in real-world scenarios.
- Not applicable if the variables are not Poisson or do not have the same distribution type.
- Limited when dealing with overdispersed data, where variance exceeds the mean significantly.

Extensions and Related Concepts

- Compound Poisson Processes: When the number of events is Poisson but each event has a random "size," leading to more complex models.
- Poisson Mixture Models: Used when the rate parameter itself is random, capturing overdispersion.
- Multivariate Poisson Distributions: Extending the idea to multiple correlated count variables.

Conclusion

The exploration of the sum of two independent Poisson random variables reveals a fundamental and elegant property: their sum is also Poisson-distributed with a parameter equal to the sum of individual parameters. This property not only simplifies analytical computations but also provides deep insights into the behavior of count data in diverse fields. Understanding the distribution of \(X

+ Y\) and the related conditional distributions allows statisticians and data scientists to model complex phenomena effectively, from network traffic to biological counts. While the Poisson distribution's simplicity and closure under addition are powerful features, practitioners must remain aware of its assumptions, particularly independence and equidispersion, to ensure accurate modeling and inference.

In summary:

- The distribution of \(X + Y\) where \(X \sim \text{Poisson}(\lambda)\) and \(Y \sim \text{Poisson}(u)\) are independent, is \(\text{Poisson}(\lambda + u)\).
- The sum's properties facilitate modeling combined event counts and understanding their distribution.
- Conditional distributions given the sum follow a binomial distribution, aiding in resource allocation and probabilistic inference.
- Recognizing these properties enhances the application of Poisson models across various disciplines, highlighting their importance in statistical theory and practice.

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