1) Solve The Following Difference Equation Using The Transform Methodz:y(k+2)+y(k)=x(k)wherex(k)is The

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Introduction to Difference Equations and the Transform Method

Difference equations are fundamental tools in discrete-time systems analysis, modeling various phenomena in fields such as control systems, signal processing, economics, and more. They describe the relationship between the current and past values of a sequence, enabling us to analyze system behavior over time.

The transform method, particularly the Z-transform, is a powerful mathematical technique used to solve linear difference equations efficiently. By converting difference equations from the time domain into the complex frequency domain, it simplifies the process of solving for the sequence $y(k)$, especially when initial conditions and input sequences are involved.

In this article, we will explore how to solve the difference equation:

$$
y(k+2) + y(k) = x(k)
$$

using the Z-transform method, assuming $x(k)$ is a known input sequence. We will detail each step, from applying the Z-transform to deriving the solution, and discuss the importance of initial conditions, inverse Z-transform, and the role of system transfer functions.

Understanding the Difference Equation

The Given Difference Equation

The equation

\[

y(k+2) + y(k) = x(k)
\]

is a linear, nonhomogeneous difference equation of second order. It relates the current value of the output sequence \(y(k) \) to its value two steps earlier, plus an input sequence \(x(k) \).

Key points:

- The order of the difference equation is 2, since the highest shift is \(k+2 \).
- The equation is linear and homogeneous if \(x(k) = 0 \); otherwise, it is nonhomogeneous.
- The input \(x(k) \) can be any known sequence, such as a step, impulse, or sinusoid.

Significance of the Equation

Solving this difference equation allows us to understand the system's response to various inputs, which is crucial in designing controllers, filters, or predicting system behavior.

Applying the Z-Transform to the Difference Equation

Definition of the Z-Transform

The Z-transform converts a discrete-time sequence \(y(k) \) into a complex function \(Y(z) \):

\[
Y(z) = \mathcal{Z}\{ y(k) \} = \sum_{k=0}^{\infty} y(k) z^{-k}
\]

It simplifies operations like shifting, convolution, and difference equations by translating them into algebraic equations.

Transforming the Difference Equation

To solve the difference equation using the Z-transform:

1. Apply the Z-transform to both sides of the equation.

2. Use properties of the Z-transform to handle shifts and initial conditions.

3. Solve algebraically for \(Y(z) \).

4. Find the inverse Z-transform to get \(y(k) \).

Step-by-Step Solution

Step 1: Apply the Z-Transform

Applying $\mathcal{Z}$ to both sides:

$$
\mathcal{Z}\{ y(k+2) \} + \mathcal{Z}\{ y(k) \} = \mathcal{Z}\{ x(k) \}
$$

Using the shift property:

$$
\mathcal{Z}\{ y(k+2) \} = z^2 Y(z) - z y(0) - y(1)
$$

and

$$
\mathcal{Z}\{ y(k) \} = Y(z)
$$

Similarly, $\mathcal{Z}\{ x(k) \} = X(z)$.

Putting it all together:

$$
z^2 Y(z) - z y(0) - y(1) + Y(z) = X(z)
$$

Step 2: Solve for $Y(z)$

Combine like terms:

$$
(z^2 + 1) Y(z) = X(z) + z y(0) + y(1)
$$

Expressing $Y(z)$:

\[

$$
Y(z) = \frac{X(z) + z y(0) + y(1)}{z^2 + 1}
$$

This is the general solution in the Z-domain, incorporating initial conditions $y(0)$ and $y(1)$.

Step 3: Determine Initial Conditions

Initial conditions are necessary to obtain a unique solution. They are usually given or assumed based on the context.

- Assumption: For illustration, assume $y(0) = y_0$ and $y(1) = y_1$.
- These values can be specified based on system state or initial measurements.

Step 4: Find the Inverse Z-Transform

The inverse Z-transform retrieves $y(k)$ from $Y(z)$:

$$
y(k) = \mathcal{Z}^{-1}\{ Y(z) \}
$$

The solution depends on the form of $X(z)$. For common inputs:

- Unit step input: $x(k) = u(k)$, $X(z) = \frac{z}{z-1}$.
- Impulse input: $x(k) = \delta(k)$, $X(z) = 1$.

Let's analyze the system's response under these typical inputs.

Example: Solving with a Step Input

Suppose $x(k) = u(k)$, the unit step function. Then:

$$
X(z) = \frac{z}{z-1}
$$

and the transfer function of the system is:

\[

$$
H(z) = \frac{1}{z^2 + 1}
$$

So, the total transfer function with input becomes:

$$
Y(z) = H(z) \times X(z) + \text{terms involving initial conditions}
$$

Explicitly:

$$
Y(z) = \frac{z}{(z-1)(z^2 + 1)} + \frac{z y(0) + y(1)}{z^2 + 1}
$$

The inverse Z-transform involves partial fraction decomposition to simplify.

Partial Fraction Decomposition and Inverse Z-Transform

To perform inverse Z-transform:

1. Decompose $Y(z)$ into simpler fractions.
2. Use standard Z-transform pairs to find $y(k)$.

For example, for the term:

$$
\frac{z}{(z-1)(z^2 + 1)}
$$

we can write:

$$
\frac{z}{(z-1)(z^2 + 1)} = \frac{A}{z-1} + \frac{Bz + C}{z^2 + 1}
$$

Solve for coefficients A, B, C, then apply inverse Z-transform to each term:

- $\mathcal{Z}^{-1}\left\{ \frac{A}{z-1} \right\} = A \cdot 1^{k} = A$
- $\mathcal{Z}^{-1}\left\{ \frac{Bz + C}{z^2 + 1} \right\}$ involves sinusoidal sequences.

Final Solution and Interpretation

The complete solution combines the homogeneous response (from initial conditions) and particular response (from the input $x(k)$). It can be summarized as:

$$
y(k) = y_h(k) + y_p(k)
$$

where:

- $y_h(k)$ is the homogeneous solution, solving the associated homogeneous difference equation $y(k+2) + y(k) = 0$.
- $y_p(k)$ is the particular solution due to the input $x(k)$.

The homogeneous solution involves characteristic roots:

$$
r^2 + 1 = 0 \Rightarrow r = \pm j
$$

leading to:

$$
y_h(k) = C_1 \cos\left(\frac{\pi}{2} k \right) + C_2 \sin\left(\frac{\pi}{2} k \right)
$$

Constants C_1, C_2 are determined by initial conditions.

The particular solution depends on the input sequence and can be obtained via the inverse Z-transform of the particular part.

Conclusion and Practical Applications

Solving difference equations using the Z-transform method provides a systematic approach to analyze discrete systems. This technique's advantages include:

- Simplification of algebraic operations in the Z-domain.
- Ease of incorporating initial conditions.
- Capability to handle complex input sequences.

Understanding this process is essential for engineers and scientists working with digital signal processing, control systems, and discrete-time system analysis.

Practical applications include:

- Designing digital filters.
- Analyzing system stability.
- Modeling economic or biological systems with discrete data.
- Developing algorithms for real-time signal processing.

In summary:

- Convert the difference equation to the Z-domain.

Frequently Asked Questions

What is the difference equation given in the problem?

The difference equation is $y(k+2) + y(k) = x(k)$.

How do you apply the Z-transform to the difference equation?

Apply the Z-transform to both sides, using properties like linearity and shifting, to convert the difference equation into an algebraic equation in the Z-domain.

What is the Z-transform of y(k+2) and y(k)?

The Z-transform of $y(k+2)$ is $z^2 Y(z) - z y(0) - y(1)$, and of $y(k)$ is $Y(z)$, assuming initial conditions $y(0)$ and $y(1)$.

How do initial conditions affect the solution when using the transform method?

Initial conditions $y(0)$ and $y(1)$ appear in the transformed equation and influence the solution for $Y(z)$, which is then inverse-transformed to find $y(k)$.

What is the Z-transform of the input x(k)?

It is $X(z)$, the Z-transform of the input sequence $x(k)$.

What algebraic equation do we obtain in the Z-domain after applying the transform?

The transformed equation becomes $z^2 Y(z) - z y(0) - y(1) + Y(z) = X(z)$.

How do you solve for Y(z) in this transformed equation?

Rearrange the algebraic equation to isolate $Y(z)$: $Y(z) = [X(z) + z y(0) + y(1)] / (z^2 + 1)$.

How is the inverse Z-transform used to find y(k)?

Apply the inverse Z-transform to Y(z) to get y(k), often using partial fraction decomposition or tables of Z-transforms.

What is the significance of the denominator (z^2 + 1) in the solution?

It indicates the characteristic polynomial of the difference equation and determines the system's stability and response characteristics.

Additional Resources

Solve The Following Difference Equation Using The Transform Method: y(k+2) + y(k) = x(k), where x(k) is the input signal

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Introduction: The Significance of Difference Equations and the Transform Method

Difference equations are mathematical expressions that relate the value of a discrete-time sequence at one or more previous points to its current or future values. These equations are fundamental in digital signal processing, control systems, economics, and various engineering disciplines because they model the dynamic behavior of systems over discrete time intervals. The ability to solve these equations efficiently is crucial for analyzing system stability, response, and behavior.

One of the most powerful techniques for solving linear difference equations is the transform method, particularly the Z-transform. This approach converts difference equations from the time domain into the complex frequency domain, simplifying the algebraic manipulation required to find explicit solutions. Once solutions are derived in the Z-domain, inverse transforms are used to interpret them back in the time domain.

This article delves into a specific second-order difference equation:
$$ y(k+2) + y(k) = x(k) $$
where $x(k)$ is an input signal, and aims to provide a comprehensive, detailed, and step-by-step solution using the Z-transform method. The discussion is structured to enhance understanding, covering all foundational aspects, key concepts, and practical applications.

Understanding the Difference Equation

Form of the Equation

The given difference equation
$$ y(k+2) + y(k) = x(k) $$
is a linear, constant-coefficient, non-homogeneous difference equation of second order.

- Order: Since the highest shift is $k+2$, it is a second-order difference equation.
- Linearity: The equation is linear in $y(k)$ and $x(k)$.
- Homogeneous part: $y(k+2) + y(k) = 0$ (when $x(k) = 0$)
- Particular solution: Depends on the specific form of $x(k)$.

Significance in Systems Theory

This equation models systems where the current output depends on its own past values and an external input. For example, in digital filters, this can represent a simple filter with feedback, where the output at a given time is influenced by previous outputs and the input signal.

Z-Transform: The Foundation of the Solution

What is the Z-Transform?

The Z-transform is a powerful tool that maps discrete sequences into a complex frequency domain.
For a sequence $y(k)$, it is defined as:
$$ Y(z) = \sum_{k=0}^{\infty} y(k) z^{-k} $$
where z is a complex variable.

Key advantages:
- Converts difference equations into algebraic equations.
- Simplifies handling initial conditions.
- Facilitates analysis of system stability and frequency response.

Properties of Z-Transform Useful for This Problem

- Linearity: $\mathcal{Z}\{a y_1(k) + b y_2(k)\} = a Y_1(z) + b Y_2(z)$
- Shift in time: $\mathcal{Z}\{ y(k + n) \} = z^{n} (Y(z) - \sum_{m=0}^{n-1} y(m) z^{-m})$
- Initial value theorem:
$$ y(0) = \lim_{z \to \infty} Y(z) $$

Step-by-Step Solution of the Difference Equation

Step 1: Apply the Z-Transform to Both Sides

Applying the Z-transform to the entire difference equation:

$$ \mathcal{Z}\{ y(k+2) \} + \mathcal{Z}\{ y(k) \} = \mathcal{Z}\{ x(k) \} $$

Using the property of the Z-transform for shifted sequences:

$$ \mathcal{Z}\{ y(k+2) \} = z^{2} Y(z) - z y(0) - y(1) $$

and

$$ \mathcal{Z}\{ y(k) \} = Y(z) $$

Similarly, the input signal transforms as:

$$ X(z) = \mathcal{Z}\{ x(k) \} $$

Putting these together, the transformed equation is:

$$ z^{2} Y(z) - z y(0) - y(1) + Y(z) = X(z) $$

Step 2: Rearrange to Solve for $Y(z)$

Group the terms:

$$ (z^{2} + 1) Y(z) = X(z) + z y(0) + y(1) $$

Expressed as:

$$ Y(z) = \frac{X(z) + z y(0) + y(1)}{z^{2} + 1} $$

This algebraic expression relates the Z-transform of the output $Y(z)$ to the input $X(z)$, initial conditions $y(0)$ and $y(1)$, and the system's characteristic polynomial $z^{2} + 1$.

Step 3: Inverse Z-Transform to Find $y(k)$

To find $y(k)$, perform the inverse Z-transform on $Y(z)$:

$$ y(k) = \mathcal{Z}^{-1} \left\{ \frac{X(z) + z y(0) + y(1)}{z^{2} + 1} \right\} $$

This step involves:

- Decomposing the right-hand side into simpler terms.
- Recognizing standard inverse transforms.
- Handling the particular solution based on $X(z)$.

Special Cases and Practical Examples

Case 1: Zero Initial Conditions and Zero Input ($y(0) = 0, y(1) = 0, x(k) = 0$)

In this simplistic scenario, the solution simplifies dramatically:

$$ Y(z) = 0 \Rightarrow y(k) = 0 $$

which aligns with the expectation that null inputs and initial conditions yield a zero output.

Case 2: Zero Initial Conditions with a Known Input $x(k)$

Suppose $x(k)$ is a known sequence, such as $x(k) = \delta(k)$ (unit impulse), then:

$$ X(z) = 1 $$

and the solution becomes:

$$ Y(z) = \frac{1}{z^{2} + 1} $$

The inverse Z-transform of this is well-known:

$$ y(k) = \sin(k) \quad \text{for} \quad x(k) = \delta(k) $$

which indicates the system's impulse response is sinusoidal, a hallmark of the second-order difference equation with characteristic roots on the imaginary axis.

Analyzing the Homogeneous Solution

The homogeneous part:

$$y(k+2) + y(k) = 0$$

has characteristic equation:

$$r^{2} + 1 = 0$$

yielding roots:

$$r = \pm i$$

The homogeneous solution:

$$y_h(k) = C_1 \cos(k) + C_2 \sin(k)$$

where C_1 and C_2 are determined by initial conditions.

This solution reflects oscillatory behavior, typical of systems with purely imaginary roots, which do not decay or grow but oscillate indefinitely.

Particular Solution and Response to $x(k)$

The particular solution depends on the form of $x(k)$:

- For constant input $x(k) = A$, the particular solution is a constant Y_p, found by substituting into the Z-domain solution.
- For sinusoidal input, the response mirrors the input's frequency, modulated by the system's transfer function.

The total solution:

$$y(k) = y_h(k) + y_p(k)$$

combines the homogeneous oscillations with the particular solution driven by $x(k)$.

Initial Conditions and Complete Solution

To fully specify the solution, initial conditions $y(0)$ and $y(1)$ are necessary. These are typically provided or measured in practical systems.

Once initial values are known, constants C_1 and C_2 are determined by solving:

$$
\begin{cases}
y(0) = C_1 \\
y(1) = C_1 \cos(1) + C_2 \sin(1) + y_p(1)
\end{cases}
$$

which ensures a unique solution aligning with the physical initial state of the system.

Conclusion: The Efficacy of the Transform Method

The transform method, particularly the Z-transform, offers a systematic approach to solving difference equations like:

$$ y(k+2) + y(k) = x(k) $$

by converting the problem into an algebraic one in the complex z-plane. This approach simplifies handling initial conditions

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