1. Use L'Hospital's Rule To Show That Lim F(x) = 0 And Lim F(x) = 0 X+00 For Planck's Law. So This Law

1. Use L'Hospital's Rule To Show That Lim F(x) = 0 And Lim F(x) = 0 X+00 For Planck's Law. So This Law

Understanding the limits involved in Planck's Law is crucial for grasping the behavior of blackbody radiation at various frequencies and temperatures. In particular, analyzing the limits of Planck's radiation function as the frequency approaches zero and infinity provides insights into the physical phenomena it models. Utilizing L'Hospital's Rule is an effective mathematical technique to evaluate these limits, especially when direct substitution results in indeterminate forms. This article explores how L'Hospital's Rule can be applied to demonstrate that the spectral radiance approaches zero as frequency tends to zero and infinity, solidifying our understanding of Planck's Law and its implications in physics and thermodynamics.

Understanding Planck's Law

Planck's Law describes the spectral radiance of a blackbody at a given temperature T as a function of frequency ν (or wavelength λ). The law is fundamental in quantum physics and thermodynamics, providing a precise model for the distribution of electromagnetic radiation emitted by a blackbody.

The mathematical form of Planck's Law in terms of frequency is:

$$
F(\nu) = \frac{8\pi h \nu^3}{c^3} \cdot \frac{1}{e^{\frac{h \nu}{kT}} - 1}
$$

where:

- $F(\nu)$ = spectral radiance per unit frequency,
- h = Planck's constant (6.626×10^{-34} Js),
- c = speed of light in vacuum (3.0×10^8 m/s),
- k = Boltzmann's constant (1.381×10^{-23} J/K),
- T = absolute temperature in Kelvin,
- ν = frequency in Hz.

Evaluating Limits of Planck's Law at Zero and Infinite Frequency

The limits of $F(\nu)$ as $\nu \to 0$ and $\nu \to \infty$ are essential in understanding the behavior of blackbody radiation:

- As $\nu \to 0$, the spectral radiance tends to zero.
- As $\nu \to \infty$, the spectral radiance also tends to zero.

To rigorously prove these limits, direct substitution results in indeterminate forms such as $0/0$ or ∞/∞. Here, L'Hospital's Rule becomes a valuable tool.

Applying L'Hospital's Rule to Limit $\lim_{\nu \to 0} F(\nu)$

Step 1: Recognize the Indeterminate Form

As $\nu \to 0$:

$$
F(\nu) = \frac{8 \pi h \nu^3}{c^3} \cdot \frac{1}{e^{\frac{h \nu}{k T}} - 1}
$$

The numerator $8 \pi h \nu^3 \to 0$, and the denominator $e^{\frac{h \nu}{k T}} - 1 \to 0$, leading to a $0/0$ form.

Step 2: Rewrite the Limit

Express the spectral radiance as a single fraction:

$$
F(\nu) = \frac{8 \pi h \nu^3}{c^3 (e^{\frac{h \nu}{k T}} - 1)}
$$

Now, consider the limit:

$$
\lim_{\nu \to 0} F(\nu) = \lim_{\nu \to 0} \frac{8 \pi h \nu^3}{c^3 (e^{\frac{h \nu}{k T}} - 1)}
$$

Since constants do not affect the limit, focus on:

$$
\lim_{\nu \to 0} \frac{\nu^3}{e^{\frac{h \nu}{k T}} - 1}
$$

Step 3: Apply L'Hospital's Rule

Because both numerator and denominator tend to zero as $\nu \to 0$, differentiate numerator and denominator separately with respect to ν:

- Numerator derivative:

$$
\frac{d}{d \nu} \left(\nu^3 \right) = 3 \nu^2
$$

- Denominator derivative:

$$
\frac{d}{d \nu} \left(e^{\frac{h \nu}{k T}} - 1 \right) = e^{\frac{h \nu}{k T}} \cdot \frac{h}{k T}
$$

Applying L'Hospital's Rule:

$$
\lim_{\nu \to 0} \frac{\nu^3}{e^{\frac{h \nu}{k T}} - 1} = \lim_{\nu \to 0} \frac{3 \nu^2}{e^{\frac{h \nu}{k T}} \cdot \frac{h}{k T}}
$$

At $\nu = 0$, $e^{0} = 1$, so the limit simplifies to:

$$
\lim_{\nu \to 0} \frac{3 \nu^2}{\frac{h}{k T}} = 0
$$

since $\nu^2 \to 0$. Therefore,

$$
\lim_{\nu \to 0} F(\nu) = 0
$$

which confirms that the spectral radiance approaches zero as frequency approaches zero.

Applying L'Hospital's Rule to Limit $\lim_{\nu \to \infty} F(\nu)$

Step 1: Recognize the Indeterminate Form

As $\nu \to \infty$:

- The numerator $8 \pi h \nu^3 \to \infty$,
- The exponential $e^{\frac{h \nu}{k T}} \to \infty$,
- The denominator $e^{\frac{h \nu}{k T}} - 1 \to \infty$.

The entire expression takes the form ∞ / ∞, suitable for L'Hospital's Rule.

Step 2: Rewrite the Limit

Focus on the main ratio:

$$
F(\nu) = \frac{8 \pi h \nu^3}{c^3 (e^{\frac{h \nu}{k T}} - 1)}
$$

which simplifies to examining:

$$
\lim_{\nu \to \infty} \frac{\nu^3}{e^{\frac{h \nu}{k T}}}
$$

since, for large ν, subtracting 1 is negligible.

Step 3: Apply L'Hospital's Rule Multiple Times

The numerator $\nu^3 \to \infty$, and the denominator $e^{\frac{h \nu}{k T}} \to \infty$. Apply L'Hospital's Rule three times:

First differentiation:

$$
\frac{d}{d \nu} (\nu^3) = 3 \nu^2
$$

$$
\frac{d}{d \nu} \left(e^{\frac{h \nu}{k T}} \right) = e^{\frac{h \nu}{k T}} \cdot \frac{h}{k T}
$$

Applying L'Hospital:

$$
\lim_{\nu \to \infty} \frac{3 \nu^2}{e^{\frac{h \nu}{k T}} \cdot \frac{h}{k T}}
$$

Second differentiation:

$$
\frac{d}{d \nu} (3 \nu^2) = 6 \nu
$$

$$
\frac{d}{d \nu} \left(e^{\frac{h \nu}{k T}} \cdot \frac{h}{k T} \right) = e^{\frac{h \nu}{k T}} \cdot \left(\frac{h}{k T} \right)^2
$$

Applying L'Hospital again:

$$
\lim_{\nu \to \infty} \frac{6 \nu}{e^{\frac{h \nu}{k T}} \cdot \left(\frac{h}{k T} \right)^2}
$$

Third differentiation:

$$
\frac{d}{d \nu} (6 \nu) = 6
$$

$$
\frac{d}{d \nu} \left(e^{\frac{h \nu}{k T}} \cdot \left(\frac{h}{k T} \right)^2 \right) = e^{\frac{h \nu}{k T}} \cdot \left(\frac{h}{k T} \right)^3
$$

Now, the limit becomes:

\[

Frequently Asked Questions

What is L'Hospital's Rule and how is it applied to analyze limits in Planck's Law?

L'Hospital's Rule states that if a limit results in an indeterminate form like 0/0 or ∞/∞, then the limit of the ratio can be found by differentiating numerator and denominator separately. In Planck's Law, it helps evaluate limits of functions involving exponential expressions as x approaches 0 or infinity.

How do we use L'Hospital's Rule to show that lim_{x→0} F(x) = 0 in Planck's Law?

By identifying the limit as an indeterminate form (e.g., 0/0), differentiating numerator and denominator with respect to x, and then evaluating the resulting limit. This process demonstrates that as x approaches 0, F(x) approaches 0.

Why does the limit of Planck's Law function tend to zero as x approaches infinity?

Because the exponential term in Planck's Law dominates, causing the numerator to grow slower than the exponential in the denominator, leading the entire function to tend to zero as x approaches infinity.

What is the significance of showing that lim_{x→0} F(x) = 0 in the context of Planck's Law?

It confirms that the spectral radiance approaches zero at very low frequencies or wavelengths, aligning with physical expectations and ensuring the law's consistency at the spectral limits.

Can you provide a step-by-step example of applying L'Hospital's Rule to limit evaluations in Planck's Law?

Yes. For example, when evaluating lim_{x→0} F(x), identify the indeterminate form, differentiate numerator and denominator with respect to x, then compute the new limit. This process explicitly shows the limit tends to zero.

What assumptions are necessary for applying L'Hospital's Rule to limits in Planck's Law?

The functions involved must be differentiable near the point of interest, and the limit must initially be an indeterminate form like 0/0 or ∞/∞. These conditions ensure the rule's validity.

How does the behavior of F(x) as x approaches infinity relate to the physical interpretation of Planck's Law?

As x approaches infinity, representing high frequencies or short wavelengths, the spectral radiance diminishes to zero, indicating that very high-energy photons are less likely, consistent with physical observations.

Is L'Hospital's Rule sufficient to evaluate all limits in Planck's Law, or are other methods necessary?

While L'Hospital's Rule is powerful for certain indeterminate forms, some limits may require algebraic manipulation, series expansions, or asymptotic analysis for evaluation, especially when multiple indeterminate forms or complex expressions are involved.

What role does exponential differentiation play in applying L'Hospital's Rule to Planck's Law limits?

Differentiating exponential terms is straightforward and often simplifies the limit evaluation, as exponentials have well-known derivatives. This facilitates applying L'Hospital's Rule to limits involving exponential functions in Planck's Law.

How does demonstrating these limits reinforce the validity of Planck's Law in physics?

Showing that the limits approach zero at spectral extremes confirms the mathematical consistency of Planck's Law with physical phenomena, such as the disappearance of radiation at very low or very high frequencies, supporting its foundational role in quantum physics.

Additional Resources

Use L'Hospital's Rule To Show That Lim F(x) = 0 And Lim F(x) = 0 as x ⍰ ⍰ for Planck's Law. So This Law

Understanding the behavior of functions as variables approach specific limits is fundamental in calculus, especially in applied contexts like physics. One such area where limit evaluation plays a critical role is in quantum physics and thermodynamics: Planck's Law. This law describes the spectral distribution of electromagnetic radiation emitted by a blackbody in thermal equilibrium. To analyze the limits of its function, particularly as the wavelength approaches zero or infinity, mathematicians and physicists often rely on tools like L'Hospital's Rule. This article aims to explore the application of L'Hospital's Rule to prove that the spectral radiance function tends to zero as the wavelength approaches zero and infinity, providing a comprehensive understanding of both the mathematical reasoning and physical implications.

Understanding Planck's Law and Its Mathematical Formulation

Planck's Law provides a quantitative description of the intensity of electromagnetic radiation emitted by a blackbody at a given temperature. The law is expressed mathematically as:

$$F(\lambda) = \frac{2hc^2}{\lambda^5} \cdot \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}$$

where:
- $F(\lambda)$ is the spectral radiance (power emitted per unit area per unit wavelength),
- h is Planck's constant,
- c is the speed of light,
- λ is the wavelength,
- k_B is Boltzmann's constant,
- T is the absolute temperature.

This function describes how the radiation intensity varies with wavelength at a fixed temperature. To analyze its limits as $\lambda \to 0$ and $\lambda \to \infty$, we need to examine the behavior of the function in these regimes.

Mathematical Challenges in Limit Evaluation

When analyzing the limits of $F(\lambda)$, certain indeterminate forms emerge:

- As $\lambda \to 0$, the exponential term $e^{\frac{hc}{\lambda k_B T}}$ tends to infinity, making the denominator large, but the numerator involves λ^{-5}. The interplay between these factors results in an indeterminate form of type $\frac{\infty}{\infty}$.

- As $\lambda \to \infty$, the exponential term $e^{\frac{hc}{\lambda k_B T}}$ approaches $e^{0} = 1$, leading the denominator to approach zero, creating an indeterminate form of type $\frac{0}{0}$.

In both cases, L'Hospital's Rule becomes a powerful tool to evaluate these limits precisely.

Applying L'Hospital's Rule to Limit as $\lambda \to 0$

Step 1: Recognize the Indeterminate Form

As $\lambda \to 0$:

- The numerator $2hc^2$ remains constant.
- The denominator involves $\lambda^5 (e^{hc/(\lambda k_B T)} - 1)$.

Since $e^{hc/(\lambda k_B T)} \to \infty$, the overall expression appears as $\frac{\text{constant}}{\infty} \to 0$, but we need to verify this rigorously because of the indeterminate form.

Step 2: Rewrite the Limit

Rearranged, the limit becomes:

$$\lim_{\lambda \to 0} F(\lambda) = \lim_{\lambda \to 0} \frac{2hc^2}{\lambda^5 (e^{hc/(\lambda k_B T)} - 1)}$$

This can be viewed as:

$$\lim_{\lambda \to 0} \frac{N(\lambda)}{D(\lambda)}$$

where:

- $N(\lambda) = 2hc^2$ (constant),
- $D(\lambda) = \lambda^5 (e^{hc/(\lambda k_B T)} - 1)$.

Because $N(\lambda)$ is constant, the limit reduces to analyzing the denominator tending to infinity. To make it more manageable, consider transforming the limit to a form suitable for L'Hospital's Rule.

Step 3: Substitution for Simplification

Set:

$$ t = \frac{hc}{\lambda k_B T} \Rightarrow \lambda = \frac{hc}{k_B T t} $$

As $\lambda \to 0$, $t \to \infty$. Rewriting the limit in terms of t:

$$ F(t) = \frac{2hc^2}{(\frac{hc}{k_B T t})^5} \cdot \frac{1}{e^{t} - 1} $$
$$ = 2hc^2 \left(\frac{k_B T t}{hc} \right)^5 \frac{1}{e^{t} - 1} $$
$$ = 2hc^2 \left(\frac{k_B T}{hc} \right)^5 t^5 \frac{1}{e^{t} - 1} $$

Now, the limit as $t \to \infty$:

$$ \lim_{t \to \infty} F(t) = \lim_{t \to \infty} C \cdot t^5 \cdot \frac{1}{e^{t} - 1} $$

where $C = 2hc^2 \left(\frac{k_B T}{hc} \right)^5$ is a constant.

Since $e^{t} \to \infty$ faster than any polynomial t^5, the entire expression tends to zero:

$$ \boxed{ \lim_{t \to \infty} F(t) = 0 } $$

Thus, as $\lambda \to 0$, $F(\lambda) \to 0$.

Note: This conclusion leverages the exponential growth in the denominator, which outpaces the polynomial numerator, a classic scenario where L'Hospital's Rule confirms the limit.

Applying L'Hospital's Rule to Limit as $\lambda \to \infty$

Step 1: Recognize the Indeterminate Form

As $\lambda \to \infty$:

- $e^{hc/(\lambda k_B T)} \to e^{0} = 1$,
- $e^{hc/(\lambda k_B T)} - 1 \to 0$,
- The numerator $2hc^2$ remains constant,
- The denominator involves $\lambda^5 \times \text{something tending to zero}$.

So, the overall form is $\frac{\text{constant}}{\infty \times 0}$, but more precisely, as the exponential approaches 1, numerator remains finite, denominator tends to zero, indicating the entire function tends to zero or perhaps infinity depending on the behavior.

Let's analyze explicitly.

Step 2: Rewrite the Limit

Express $F(\lambda)$ as:

$$ F(\lambda) = \frac{2hc^2}{\lambda^5} \times \frac{1}{e^{hc/(\lambda k_B T)} - 1} $$

As $\lambda \to \infty$, $e^{hc/(\lambda k_B T)} \to 1$, so:

$$ e^{hc/(\lambda k_B T)} - 1 \to 0 $$

and the entire function resembles $\frac{\text{finite constant}}{\infty \times 0}$.

To evaluate the limit, observe that:

$$\lim_{\lambda \to \infty} F(\lambda) = 0$$

because the numerator tends to zero (since $1/\lambda^5 \to 0$), and the denominator tends to zero as well, but the numerator's decay dominates, leading to the conclusion that:

$$\boxed{ \lim_{\lambda \to \infty} F(\lambda) = 0 }$$

Alternatively, for rigorous proof, set $t = hc/(\lambda k_B T)$. As $\lambda \to \infty$, $t \to 0$. Then:

$$F(t) = 2hc^2 \left(\frac{k_B T}{hc} \right)^5 t^5 \frac{1}{e^{t} - 1}$$

Now, as $t \to 0$:

$$e^{t} - 1 \sim t + \frac{t^2}{2} + \dots$$

Thus:

$$\frac{t^5}{e^{t} - 1} \sim \frac{t^5}{t} = t^4 \to 0$$

Therefore, $F(t) \to 0$, confirming the limit as $\lambda \to \infty$.

Physical Significance of the Limits

The mathematical limits derived using L'Hospital's Rule are not just

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