11. 10 Points In R4 With The Standard Inner Product, (a) Find A Basis For The Orthogonal Complement Wt

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Understanding the concept of orthogonal complements in vector spaces, especially in $\mathbb{R}^4$, is fundamental to linear algebra and its applications. In this article, we will thoroughly explore how to find a basis for the orthogonal complement $W^{\perp}$ of a given subspace W in $\mathbb{R}^4$ using the standard inner product. We will break down the process step-by-step, providing clear explanations, illustrative examples, and practical tips to enhance your understanding.

Introduction to Orthogonal Complements in $\mathbb{R}^4$

What Is an Orthogonal Complement?

In the context of vector spaces, the orthogonal complement of a subspace $W \subseteq \mathbb{R}^4$ is the set of all vectors in $\mathbb{R}^4$ that are orthogonal to every vector in W. Formally, it is defined as:

$$
W^{\perp} = \{ \mathbf{v} \in \mathbb{R}^4 \mid \mathbf{v} \cdot \mathbf{w} = 0, \, \forall \mathbf{w} \in W \}
$$

where $\mathbf{v} \cdot \mathbf{w}$ is the standard inner product in $\mathbb{R}^4$.

Significance of Orthogonal Complements

- They help in decomposing a vector space into direct sums: $\mathbb{R}^4 = W \oplus W^{\perp}$.
- They are crucial in solving systems of linear equations, least squares problems, and in projection methods.
- Understanding these complements aids in grasping dual spaces and orthogonal

projections, which are foundational in both theoretical and applied mathematics.

Preliminaries and Key Concepts

Standard Inner Product in $\mathbb{R}^4$

The standard inner product (dot product) in $\mathbb{R}^4$ for vectors $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4)$ is given by:

$$
\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4
$$

This operation measures the "angle" and "lengths" of vectors, facilitating the concept of orthogonality: two vectors are orthogonal if their inner product is zero.

Subspaces in $\mathbb{R}^4$

Subspaces are subsets of $\mathbb{R}^4$ closed under vector addition and scalar multiplication. Examples include:

- The zero vector space $\{ \mathbf{0} \}$.
- The span of a set of vectors.
- Hyperplanes defined by linear equations.

To find the orthogonal complement $W^{\perp}$, we need to understand the structure of W.

Basis and Dimension

- A basis of a subspace W is a minimal set of vectors that span W.
- The dimension of W, denoted $\dim W$, is the number of vectors in its basis.
- The orthogonal complement $W^{\perp}$ is itself a subspace, and its dimension satisfies:

$$
\dim W + \dim W^{\perp} = 4
$$

\]

since \(\mathbb{R}^4\) has dimension 4.

Step-by-Step Procedure to Find a Basis for \(W^{\perp}\)

Suppose you are given a subspace \(W \subseteq \mathbb{R}^4\), typically described by a set of vectors or equations. The goal is to identify a basis for \(W^{\perp}\).

Step 1: Identify the Subspace \(W\)

- Typically, \(W\) is given as the span of some vectors:

\[
W = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}
\]

- Alternatively, \(W\) might be described by a set of linear equations:

\[
A \mathbf{x} = \mathbf{0}
\]

where \(A\) is a matrix with rows representing the equations defining \(W\).

Step 2: Find a Basis for \(W\) (if necessary)

- If \(W\) is given via generators, ensure they are linearly independent.
- Use Gaussian elimination or other methods to obtain a basis for \(W\).

Step 3: Set Up Orthogonality Conditions

- The vectors in \(W^{\perp}\) are orthogonal to all basis vectors of \(W\).
- For each basis vector \(\mathbf{w}_i\), the orthogonality condition is:

\[
\mathbf{v} \cdot \mathbf{w}_i = 0
\]

- If W is defined by equations, solving these equations directly yields $W^{\perp}$.

Step 4: Formulate the System of Equations

- Collect the basis vectors of W into a matrix:

$$
\begin{bmatrix}
w_{11} & w_{12} & w_{13} & w_{14} \\
w_{21} & w_{22} & w_{23} & w_{24} \\
\vdots & \vdots & \vdots & \vdots \\
w_{k1} & w_{k2} & w_{k3} & w_{k4}
\end{bmatrix}
$$

- The orthogonality conditions form a homogeneous system:

$$
\begin{bmatrix}
w_{11} & w_{12} & w_{13} & w_{14} \\
w_{21} & w_{22} & w_{23} & w_{24} \\
\vdots & \vdots & \vdots & \vdots \\
w_{k1} & w_{k2} & w_{k3} & w_{k4}
\end{bmatrix}
\begin{bmatrix}
v_1 \\ v_2 \\ v_3 \\ v_4
\end{bmatrix}
= \mathbf{0}
$$

- Solving this system provides the vectors in $W^{\perp}$.

Step 5: Solve the System and Find the Basis

- Use Gaussian elimination to reduce the matrix to row-echelon form.
- Find the general solution, expressed in parametric form.
- The set of vectors corresponding to the free variables forms a basis for $W^{\perp}$.

Step 6: Verify the Basis Vectors

- Confirm that the resulting vectors are orthogonal to all vectors in W.
- Check that these vectors are linearly independent.
- Confirm that the number of basis vectors matches $\dim W^{\perp}$.

Illustrative Example: Finding $W^{\perp}$ in $\mathbb{R}^4$

Suppose W is spanned by two vectors:

$$
\mathbf{w}_1 = (1, 2, 0, -1), \quad \mathbf{w}_2 = (0, 1, 3, 2)
$$

Our goal is to find a basis for $W^{\perp}$.

Step 1: Form the matrix with the basis vectors as rows

$$
M = \begin{bmatrix}
1 & 2 & 0 & -1 \\
0 & 1 & 3 & 2
\end{bmatrix}
$$

Step 2: Set up the orthogonality equations

Let $\mathbf{v} = (v_1, v_2, v_3, v_4)$. The system:

$$
\begin{cases}
1 \cdot v_1 + 2 \cdot v_2 + 0 \cdot v_3 - 1 \cdot v_4 = 0 \\
0 \cdot v_1 + 1 \cdot v_2 + 3 \cdot v_3 + 2 \cdot v_4 = 0
\end{cases}
$$

Simplifies to:

$$
\begin{cases}
v_1 + 2 v_2 - v_4 = 0 \\
v_2 + 3 v_3 + 2 v_4 = 0
$$

Step 3: Express leading variables in terms of free variables

Frequently Asked Questions

What is the main goal when finding the orthogonal complement W⊥ in R^4 with the standard inner product?

The main goal is to find all vectors in R^4 that are orthogonal to every vector in the subspace W, effectively determining its orthogonal complement W⊥.

How do you find a basis for the orthogonal complement W⊥ in R^4?

To find a basis for W⊥, set up equations based on the condition that vectors are orthogonal to all vectors in W, then solve these equations to find the general form of vectors in W⊥, and extract basis vectors.

What role does the standard inner product play in determining W⊥ in R^4?

The standard inner product defines the orthogonality condition: two vectors are orthogonal if their dot product is zero. This condition is used to set up the equations for W⊥.

Can you give an example of how to find W⊥ when W is spanned by specific vectors in R^4?

Yes. If W is spanned by vectors v_1, v_2, ..., then W⊥ consists of all vectors x in R^4 such that $x \cdot v_1 = 0$, $x \cdot v_2 = 0$, etc. Solving these equations yields the basis for W⊥.

What is the significance of the dimension theorem regarding W and W⊥ in R^4?

The theorem states that dim(W) + dim(W⊥) = 4 in R^4, which helps verify the correctness of the basis for W⊥ once the dimension of W is known.

How do you verify that your found basis vectors for W⊥ are correct?

You verify by checking that each basis vector is orthogonal to all vectors spanning W and that the basis vectors are linearly independent.

What are common mistakes to avoid when finding the basis for W⊥ in R^4?

Common mistakes include forgetting to include all orthogonality conditions, making algebraic errors in solving the system, or assuming the wrong dimension for W, leading to an incorrect basis for W⊥.

Additional Resources

10 Points in R4 with the Standard Inner Product: Find a Basis for the Orthogonal Complement W⊥

Understanding the geometry of subspaces within $\mathbb{R}^4$ is fundamental in linear algebra, especially when exploring the relationships between a subspace W and its orthogonal complement $W^\perp$. The problem of finding a basis for $W^\perp$ given a set of points (vectors) in $\mathbb{R}^4$ is a classic exercise that combines concepts of linear independence, orthogonality, and subspace dimension. In this guide, we will delve into 10 points in $\mathbb{R}^4$ with the standard inner product and demonstrate how to find a basis for the orthogonal complement $W^\perp$, providing a step-by-step approach that is both comprehensive and accessible.

Introduction to the Problem

Suppose you are given a set of vectors in $\mathbb{R}^4$ that define a subspace W. The goal is to find a basis for the orthogonal complement $W^\perp$, which consists of all vectors in $\mathbb{R}^4$ that are orthogonal to every vector in W. This process involves:

- **Understanding the standard inner product in $\mathbb{R}^4$, which is defined as**

$$
\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4
$$

- Using the properties of orthogonality to derive the defining equations for $W^\perp$.
- Computing the basis for $W^\perp$ through linear algebra techniques such as row reduction.

Step 1: Clarify the Given Set of Points and the Subspace W

The Set of Points

Typically, "points in $\mathbb{R}^4$" refer to vectors like:

$$
\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k
$$

Suppose we are given 10 points:

$$
\begin{aligned}
\mathbf{v}_1 &= (v_{11}, v_{12}, v_{13}, v_{14}) \\
\mathbf{v}_2 &= (v_{21}, v_{22}, v_{23}, v_{24}) \\
&\vdots \\
\mathbf{v}_{10} &= (v_{101}, v_{102}, v_{103}, v_{104})
\end{aligned}
$$

Defining the Subspace W

The subspace W can be defined as the span of these points:

$$
W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{10}\}
$$

If these vectors are linearly dependent, then the actual dimension of W will be less than 10. To proceed, we typically:

- Collect the vectors into a matrix V where each row is a vector.
- Perform row operations to determine the rank and find a basis for W.

Step 2: Find a Basis for W

Constructing the Matrix

Arrange the vectors as rows in a matrix:

$$
V = \begin{bmatrix}
v_{11} & v_{12} & v_{13} & v_{14} \\
v_{21} & v_{22} & v_{23} & v_{24} \\
\vdots & \vdots & \vdots & \vdots \\
v_{101} & v_{102} & v_{103} & v_{104}
\end{bmatrix}
$$

\]

Row Reduce to Find the Rank

Perform Gaussian elimination to:

- Identify the pivot positions.
- Extract the linearly independent vectors as basis vectors for \(W\).

Note: If the vectors are already linearly independent, then all 10 are part of the basis, but typically, the maximum independent set will be less than or equal to 4 (since the space is \(\mathbb{R}^4\)).

Example

Suppose after reduction, the rank of \(V\) is 3, indicating \(W\) is 3-dimensional. The basis vectors are the original vectors corresponding to the pivot rows.

Step 3: Find the Orthogonal Complement \(W^\perp\)

Theoretical Foundation

The orthogonal complement \(W^\perp\) consists of all vectors \(\mathbf{x} \in \mathbb{R}^4\) such that:

\[

\langle \mathbf{x}, \mathbf{w} \rangle = 0, \quad
\forall \mathbf{w} \in W
\]

If the basis vectors of \(W\) are \(\mathbf{w}_1,
\mathbf{w}_2, \dots, \mathbf{w}_k\), then:

\[
W^\perp = \left\{\mathbf{x} \in \mathbb{R}^4 \mid
\langle \mathbf{x}, \mathbf{w}_i \rangle = 0, \quad
i=1,\dots,k \right\}
\]

This set of equations forms a homogeneous system of
linear equations.

Step 4: Set Up and Solve the System of Equations

Formulate the System

Construct the matrix \(A\) whose rows are the basis
vectors of \(W\):

\[
A = \begin{bmatrix}
w_{11} & w_{12} & w_{13} & w_{14} \\
w_{21} & w_{22} & w_{23} & w_{24} \\
\vdots & \vdots & \vdots & \vdots \\
w_{k1} & w_{k2} & w_{k3} & w_{k4}
\end{bmatrix}
\]

The orthogonal complement is given by the solutions to:

$$
A \mathbf{x} = \mathbf{0}
$$

where:

$$
\mathbf{x} = (x_1, x_2, x_3, x_4)^T
$$

Row Reduce A

- Perform Gaussian elimination on A to find its null space.
- The solutions to this homogeneous system form the basis for $W^\perp$.

Step 5: Find the Basis for $W^\perp$

Extract the Solutions

From the row-reduced form, identify the free variables and express the basic variables in terms of these free variables.

Write the General Solution

Express $\mathbf{x}$ as a linear combination of vectors corresponding to each free variable, forming the basis vectors for $W^\perp$.

Example

Suppose after reduction, the solution set is:

$$
x_1 = 2s + t, \quad x_2 = -s + 3t, \quad x_3 = s, \quad x_4 = t
$$

for free parameters s, t. Then, the basis vectors are:

$$
\begin{aligned}
\mathbf{v}_1 &= (2, -1, 1, 0) \\
\mathbf{v}_2 &= (1, 3, 0, 1)
\end{aligned}
$$

and these vectors form a basis for $W^\perp$.

Practical Example: Putting It All Together

Given Vectors

Suppose the points in $\mathbb{R}^4$ are:

$$
\begin{aligned}
\mathbf{v}_1 &= (1, 0, 0, 0) \\
\end{aligned}
$$

\mathbf{v}_2 &= (0, 1, 0, 0) \\
\mathbf{v}_3 &= (0, 0, 1, 0) \\
\mathbf{v}_4 &= (0, 0, 0, 1) \\
\mathbf{v}_5 &= (1, 1, 1, 1) \\
\text{(and so on, up to 10 points)}
\end{aligned}
\]

In this case, the first four vectors are standard basis vectors, spanning \(\mathbb{R}^4\). The remaining vectors are linear combinations. The basis for \(W\) can be taken from the linearly independent vectors identified, and then the process above is applied to find \(W^\perp\).

Additional Tips and Considerations

- Linear Dependence: Before proceeding, always check for linear dependence among the vectors. Redundant vectors do not contribute to the basis.
- Dimension Checks: Recall that in \(\mathbb{R}^4\), the sum of the dimensions of \(W\) and \(W^\perp\) equals 4:

\[
\dim(W) + \dim(W^\perp) = 4
\]

Use this to verify your results.
- Computational Tools: Use software like MATLAB, Python (NumPy), or online calculators for row reduction and null space computations for

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