(a) A UMVU Estimator Is Unique If T0, And T1 Are Both UMVU Estimators. Show That T0 = T1 Almost Surely

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Introduction to UMVU Estimators and their Importance

The concept of unbiasedness, sufficiency, and minimum variance plays a pivotal role in statistical estimation. Among estimators that satisfy these properties, the Uniformly Minimum Variance Unbiased (UMVU) estimator holds a distinguished position. A UMVU estimator is an unbiased estimator that has the lowest variance among all unbiased estimators for a parameter, uniformly over the entire parameter space. Its uniqueness property is fundamental because it guarantees that, under certain conditions, the UMVU estimator is well-defined and canonical, ensuring consistency and interpretability of statistical inference.

This article aims to demonstrate a core property of UMVU estimators: if two estimators are both UMVU estimators of the same parameter, then they must be equal almost surely. This result ensures the uniqueness of the UMVU estimator within its class, reinforcing its role as a canonical choice for estimation.

Preliminaries and Notation

Suppose we have a statistical model with sample space $\mathcal{X}$, parameter space Θ, and a family of probability measures $\{P_\theta : \theta \in \Theta\}$. Let T_0 and T_1 be two estimators of a parameter $\tau(\theta)$ based on the data $X \in \mathcal{X}$.

- Both T_0 and T_1 are assumed to be unbiased:
$$
\mathbb{E}_\theta[T_i] = \tau(\theta), \quad \forall \theta \in \Theta, \quad i=0,1.
$$
- Both are UMVU estimators, meaning:
$$
\operatorname{Var}_\theta(T_i) \leq \operatorname{Var}_\theta(S), \quad \forall \theta \in \Theta,
$$
for any other unbiased estimator S.

The goal is to show:
$$
P_\theta(\{ x \in \mathcal{X} : T_0(x) \neq T_1(x) \}) = 0,
$$
i.e., T_0 and T_1 are equal almost surely under every P_θ, which implies their equality almost everywhere with respect to the

distribution.

Understanding the Core Idea of the Proof

The proof hinges on the properties of the estimators' variances and the fact that both are minimal in variance among unbiased estimators. Intuitively, if both are UMVU, then neither can be "improved" upon in terms of variance, which constrains their possible discrepancies.

Key steps include:
- Utilizing the fact that the difference between two unbiased estimators has zero mean.
- Applying variance calculations to the difference.
- Showing that the variance of the difference must be zero, leading to the conclusion that the difference is zero almost surely.

Step-by-Step Derivation

Step 1: Consider the Difference of the Estimators

Let $D = T_0 - T_1$. Since both are unbiased estimators:
$$
\mathbb{E}_\theta[D] = \mathbb{E}_\theta[T_0] - \mathbb{E}_\theta[T_1] = \tau(\theta) - \tau(\theta) = 0, \quad \forall \theta \in \Theta.
$$

This implies D is an unbiased estimator of zero.

Step 2: Variance of the Difference

Calculate the variance of D:
$$
\operatorname{Var}_\theta(D) = \operatorname{Var}_\theta(T_0 - T_1) = \operatorname{Var}_\theta(T_0) + \operatorname{Var}_\theta(T_1) - 2 \operatorname{Cov}_\theta(T_0, T_1).
$$

Since both T_0 and T_1 are UMVU, they are both minimal variance unbiased estimators, and in particular:
$$
\operatorname{Var}_\theta(T_0) \leq \operatorname{Var}_\theta(S), \quad \text{for any unbiased } S,
$$
and similarly for T_1.

Step 3: Variance Inequalities and the Covariance

Because the variance of the difference D is non-negative:
$$
\operatorname{Var}_\theta(D) \geq 0,
$$
and the covariance term satisfies:
$$
\operatorname{Cov}_\theta(T_0, T_1) \leq \sqrt{\operatorname{Var}_\theta(T_0) \operatorname{Var}_\theta(T_1)},
$$

by the Cauchy-Schwarz inequality.

Suppose, for contradiction, that $T_0 \neq T_1$ with positive probability under some P_θ. Then, the variance of the difference D would be strictly positive.

Step 4: Showing the Variance of the Difference Is Zero

As both are UMVU estimators, they are "best" in their class, and the variance of their difference must be zero for the estimators to be consistent and the minimal variance property to hold simultaneously. Formally:

- Because T_0 and T_1 are both UMVU, any difference D which is unbiased for zero must have variance zero:

$$
\operatorname{Var}_\theta(D) = 0, \quad \forall \theta \in \Theta.
$$

This conclusion follows because if the variance were positive, one could construct an estimator with less variance than the minimal variance estimator, contradicting the UMVU property.

Step 5: Almost Sure Equality of the Estimators

Since $\operatorname{Var}_\theta(D) = 0$ for all θ, the random variable D must be constant P_θ-almost surely. But the expected value of D is zero, so the constant must be zero:

$$
D = T_0 - T_1 = 0 \quad \text{almost surely under } P_\theta, \quad \forall \theta \in \Theta.
$$

Thus, for all θ, the two estimators are equal P_θ-almost surely:

$$
P_\theta(\{ x : T_0(x) \neq T_1(x) \}) = 0.
$$

Implications and Significance of the Result

This proof demonstrates the uniqueness of the UMVU estimator within the class of unbiased estimators. It underscores the fact that if two estimators are both UMVU estimators of the same parameter, they must coincide with probability one under the model, making the UMVU estimator essentially unique.

The implications are profound:

- Canonical Estimation: The UMVU estimator serves as a canonical, well-defined estimator for the parameter, avoiding ambiguity.
- Optimality and Consistency: The uniqueness emphasizes the optimality of the UMVU estimator, providing confidence in its use.
- Theoretical Clarity: This result clarifies the structure of the space of unbiased estimators, showing that the UMVU estimator forms a singleton within this space.

Extensions and Related Results

The proof hinges on the properties of variance and unbiasedness, but similar logic applies in more complex settings, such as:

- Estimators in infinite-dimensional spaces (e.g., function spaces).
- Estimation under constraints or additional structure.
- Bayesian estimators that incorporate prior information.

Moreover, the concept of minimal variance and uniqueness extends to other optimality criteria, such as the Best Linear Unbiased Estimator (BLUE) in linear models.

Conclusion

In summary, the proof that two UMVU estimators of the same parameter are equal almost surely consolidates the foundational understanding of estimator optimality in statistical inference. It affirms that under the classical unbiasedness and variance minimization principles, the UMVU estimator is not just optimal but also unique in its class. This property ensures that statistical procedures relying on the UMVU estimator are well-defined, consistent, and interpretable, reinforcing its critical role in the theory of statistical estimation.

Frequently Asked Questions

What does it mean for an estimator to be UMVU?

An estimator is UMVU (Uniformly Minimum Variance Unbiased) if it is unbiased and has the lowest variance among all unbiased estimators for a parameter, uniformly over all parameter values.

Under what conditions are two UMVU estimators considered to be almost surely equal?

If both estimators are UMVU for the same parameter and are measurable functions of a complete sufficient statistic, then they are almost surely equal, due to the uniqueness property of UMVU estimators.

Why does the completeness of the sufficient statistic T ensure the uniqueness of the UMVU estimator?

Completeness ensures that any unbiased estimator that is a function of T is uniquely determined because the only unbiased functions with zero expectation for all parameter values are almost surely zero, leading to the uniqueness of the UMVU estimator.

How does the Lehmann-Scheffé theorem relate to the uniqueness of UMVU estimators?

The Lehmann-Scheffé theorem states that the UMVU estimator is the unique unbiased estimator that is a function of a complete sufficient statistic, ensuring its uniqueness almost surely.

What is the significance of the statement 'T0 = T1 almost surely' when both are UMVU estimators?

It signifies that, with probability one, the two estimators are equal across all sample outcomes, confirming their almost sure equivalence and uniqueness as the UMVU estimator.

Can two different functions of the same complete sufficient statistic be UMVU estimators? Why or why not?

No, because any two unbiased estimators that are functions of the same complete sufficient statistic and are both UMVU must be almost surely equal, due to the properties of completeness and the Lehmann-Scheffé theorem.

What is the main implication of the statement 'T0 = T1 almost surely' for practical statistical estimation?

It implies that, in practice, there is essentially a unique UMVU estimator for the parameter, and any other unbiased estimator that is a function of the same complete sufficient statistic will coincide with it almost surely.

Additional Resources

Unbiased Minimum Variance Uniquely Identified: The Proof of Uniqueness of UMVU Estimators

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Introduction

In the realm of statistical estimation, the quest for an estimator that judiciously balances bias and variance is perennial. Among the various classes of estimators, the Uniformly Minimum Variance Unbiased (UMVU) estimator holds a distinguished position owing to its optimality within the family of unbiased estimators. The defining characteristic of a UMVU estimator is its minimal variance among all unbiased estimators for a given parameter. However, an intriguing question arises in statistical theory: Is the UMVU estimator unique? More precisely, if two estimators, say T_0 and T_1, are both UMVU estimators for the same parameter, can they differ? The answer, rooted in the properties of sufficiency, completeness, and the Lehmann-Scheffé theorem, is that they must be equal almost surely. This profound result ensures that the UMVU estimator, when it exists, is essentially unique.

This article meticulously explores the conditions under which the UMVU estimator is unique, elucidating the underlying theoretical foundations, and providing a detailed proof that $T_0 = T_1$ almost surely when both are UMVU estimators. We will traverse through the concepts of sufficiency, completeness, the Lehmann-Scheffé theorem, and the implications of these principles for estimator uniqueness.

Theoretical Foundations of UMVU Estimators

What Is an UMVU Estimator?

In statistical inference, an estimator is a rule or function $T(X)$ that provides an estimate of a parameter θ based on observed data X. An estimator T is unbiased if its expected value equals the true parameter value for all θ in the parameter space:

$$
E_\theta[T(X)] = \theta, \quad \forall \theta
$$

A UMVU estimator is an unbiased estimator that attains the lowest possible variance among all unbiased estimators for the same parameter. Formally, T is UMVU for θ if:

$$
\operatorname{Var}_\theta[T^(X)] \leq \operatorname{Var}_\theta[T(X)], \quad \text{for all unbiased estimators } T
$$

The existence of a UMVU estimator is often guaranteed under specific conditions, particularly when sufficient and complete statistics are involved.

The Role of Sufficient and Complete Statistics

Sufficiency pertains to the data reduction principle: a sufficient statistic $T(X)$ captures all the information about the parameter θ contained in the data X. Formally, $T(X)$ is sufficient for θ if the conditional distribution of X given $T(X)$ does not depend on θ.

Completeness is a stronger condition. A sufficient statistic $T(X)$ is complete if, for any measurable function g,

$$
E_\theta[g(T(X))] = 0, \quad \forall \theta \implies P(g(T(X))=0) = 1
$$

In essence, no non-trivial function of the statistic has an expected value of zero for all θ unless it is almost surely zero. Completeness is crucial for the uniqueness of unbiased estimators because, combined with sufficiency, it allows the application of the Lehmann–Scheffé theorem to guarantee the existence and uniqueness of the UMVU estimator.

The Core Theorem: Uniqueness of the UMVU Estimator

Statement of the Theorem

Theorem:
Suppose T_0 and T_1 are both UMVU estimators for a parameter θ. If T_0 and T_1 are estimators based on a complete sufficient statistic $T(X)$, then $T_0 = T_1$ almost surely.

This theorem essentially states that the UMVU estimator, when it exists, is unique up to almost sure equality. The key assumptions are the existence of a complete sufficient statistic and the fact that both T_0 and T_1 are unbiased estimators that achieve the minimum variance.

Significance of the Theorem

The theorem's implications are profound in statistical theory:

- Estimator Uniqueness: It guarantees that the UMVU estimator is uniquely determined (except on a set of probability zero).
- Practical Implication: When two candidate estimators are both UMVU, they must be indistinguishable almost surely, simplifying the selection process.
- Theoretical Rigor: It reinforces the importance of sufficient and complete statistics in constructing uniquely optimal estimators.

Detailed Proof of the Theorem

Assumptions

- T_0 and T_1 are both unbiased estimators of θ.
- Both estimators are functions of the same complete sufficient statistic $T(X)$.
- The estimators are measurable functions of $T(X)$:

$$
T_{0} = g_{0}(T(X)), \quad T_{1} = g_{1}(T(X))
$$

- Both T_0 and T_1 attain the minimum variance among all unbiased estimators (i.e., they are UMVU).

Step-by-Step Proof

1. Recognize Both Estimators as Functions of a Complete Sufficient Statistic

Since T_0 and T_1 are UMVU estimators, the Lehmann-Scheffé theorem states that they can be expressed as functions of the complete sufficient statistic $T(X)$:

$$
T_{0} = g_{0}(T), \quad T_{1} = g_{1}(T)
$$

for some measurable functions g_{0} and g_{1}.

2. Define the Difference Function

Construct the function:

$$
h(T) = g_{0}(T) - g_{1}(T)
$$

which satisfies:

$$
E_\theta[h(T)] = E_\theta[g_{0}(T)] - E_\theta[g_{1}(T)] = \theta - \theta = 0, \quad \forall \theta
$$

because both T_0 and T_1 are unbiased estimators.

3. Show that $h(T)$ Has Zero Expectation for All θ

By the previous step:

$$
E_\theta[h(T)] = 0, \quad \forall \theta
$$

This property holds for the function $h(T)$.

4. Apply the Completeness of T to $h(T)$

Since T is complete, the only measurable functions with zero expectation for all θ are functions that are zero almost surely:

$$
P(h(T) = 0) = 1
$$

which implies:

$$
g_{0}(T) = g_{1}(T) \quad \text{almost surely}
$$

and consequently:

$$
T_{0} = T_{1} \quad \text{almost surely}
$$

This completes the proof.

Practical Implications and Examples

Implication in Estimation Procedures

The proof underscores the fact that, under the conditions of sufficiency and completeness, the UMVU estimator is unique. This simplifies the process of estimator selection in applied settings because once a UMVU estimator is identified, there's no ambiguity about its form.

Example: Estimating the Mean of a Normal Distribution

Suppose $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$ i.i.d., with known σ^2. The sample mean $\bar{X}$ is a sufficient and complete statistic for μ. The UMVU estimator for μ is $\bar{X}$ itself.

- If someone proposes another unbiased estimator T_1, based on $\bar{X}$, then by the theorem, $T_1 = \bar{X}$ almost surely.
- Any deviation would violate the uniqueness, assuming the estimator is unbiased and based on the complete sufficient statistic.

Limitations and Conditions

While the theorem guarantees uniqueness under the stated conditions, in practice:

- Complete sufficient statistics may not always exist.
- The existence of a UMVU estimator isn't guaranteed in all models.
- When the UMVU estimator exists, it is uniquely determined up to almost sure equality.

Broader Context and Related Results

Connections with the Lehmann-Scheffé Theorem

The Lehmann-Scheffé theorem states that the best unbiased estimator (minimum variance unbiased estimator) can be obtained by taking the conditional expectation of an unbiased estimator with respect to a sufficient and complete statistic. The current result builds on this theorem, emphasizing the uniqueness of such an estimator.

Extension to Other Estimator Classes

While the discussion centers around unbiased estimators, similar principles can be extended or contrasted with biased estimators or admissibility considerations, where uniqueness may not hold.

Conclusion

The question of whether the UMVU estimator is unique is fundamental in statistical theory, with practical consequences for estimation procedures. The core result—that two UMVU estimators based on a complete sufficient statistic must be equal almost surely—relies on the properties of sufficiency, completeness, and the structure of unbiased estimators.

This proof and understanding provide a solid foundation for statisticians to confidently identify and rely on the UMVU estimator once the necessary conditions are met. It assures that, within the specified framework, the pursuit of the "best" unbiased estimator leads to a uniquely defined solution, solidifying the theoretical underpinnings of optimal estimation.

By internalizing these principles, researchers and practitioners can better appreciate the elegance and rigor of statistical estimation, ensuring their methods

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