Use Cylindrical Coordinates Find The Volume Of The Solid That Is Enclosed By The Cone Z = X+y And The

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Introduction

Understanding the volume of solids bounded by surfaces is a fundamental problem in multivariable calculus. When the surfaces involved are symmetric or have a specific geometric form, choosing the appropriate coordinate system simplifies the problem significantly. Cylindrical coordinates are particularly useful when dealing with surfaces involving circular symmetry or cone-shaped regions. In this article, we explore how to find the volume of a solid enclosed by the cone $z = x + y$ using cylindrical coordinates. Although the problem statement appears incomplete, we will assume the other boundary is a plane or surface that intersects with the cone, allowing us to define a bounded region for volume calculation.

Understanding the Surfaces and Coordinates

Definition of Cylindrical Coordinates

Cylindrical coordinates (r, θ, z) are related to Cartesian coordinates (x, y, z) via:
- $x = r \cos \theta$
- $y = r \sin \theta$
- $z = z$

This coordinate system is especially suitable for regions with rotational symmetry about the z-axis. The Jacobian determinant for the transformation from Cartesian to cylindrical coordinates is:

$$dV = r \, dr \, d\theta \, dz$$

Surface Equations in Cylindrical Coordinates

The key surfaces involved are:

- The cone: $z = x + y$. In cylindrical coordinates:

$$z = r \cos \theta + r \sin \theta = r (\cos \theta + \sin \theta)$$

\]

- The other boundary (assumed to be a plane or surface), which we will specify later based on the problem context.

Setting Up the Problem

Identifying the Bounded Region

To compute the volume enclosed by the cone $z = x + y$ and the other boundary, we need to:

1. Define the region in the xy-plane over which the solid extends.
2. Express the upper and lower bounds of z in terms of r and θ.

Suppose the other boundary is the plane $z = 0$, which is common in volume problems involving cones, but the problem statement suggests an incomplete boundary description. For illustrative purposes, let's assume the bounded region is between:

- The cone $z = r (\cos \theta + \sin \theta)$,
- The plane $z = 0$,
- And the region in the xy-plane over which the volume is enclosed.

In many cone-volume problems, the boundary in the xy-plane is determined by the intersection of the cone with a circle or sector. Assume the region is bounded by the cone and the xy-plane, and the projection onto the xy-plane is a sector in the xy-plane.

Finding the Intersection and Boundaries

Determining the Projection Region

The volume is bounded below by $z=0$ and above by $z = r (\cos \theta + \sin \theta)$. The projection onto the xy-plane involves the region where this surface is non-negative:

$$
z \geq 0 \Rightarrow r (\cos \theta + \sin \theta) \geq 0
$$

This inequality defines the angular region for θ:
- When $\cos \theta + \sin \theta \geq 0$, the region exists.
- When $\cos \theta + \sin \theta < 0$, the cone dips below the xy-plane, which we exclude if we're considering the volume above $z=0$.

Since $\cos \theta + \sin \theta \geq 0$ simplifies to:
\[
\cos \theta + \sin \theta \geq 0

\]
which can be rewritten as:
$$
\sqrt{2} \sin \left(\theta + \frac{\pi}{4} \right) \geq 0
$$
or
$$
\sin \left(\theta + \frac{\pi}{4} \right) \geq 0
$$
Thus, the angular region of integration is:
$$
- \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}
$$

Radial Boundaries

The radial boundary depends on where the surface intersects the xy-plane, which is at $z=0$:
$$
z = r (\cos \theta + \sin \theta) = 0
$$
This occurs when either:
- $r=0$, or
- $\cos \theta + \sin \theta = 0$, which is consistent with the boundary conditions already determined.

The maximum r, for a fixed θ, depends on the extent of the region. If the boundary is a circle of radius R, or the cone is truncated at some height Z, the bounds can be adjusted accordingly.

For simplicity, assume the cone extends up to a certain height Z, or the boundary is at a fixed radius R. For a general approach, the limits are:

- $0 \leq r \leq R(\theta)$, where $R(\theta)$ is the maximum radius in the θ direction before reaching the boundary.

Calculating the Volume

Setting Up the Double Integral

The volume V is computed by integrating the height z over the projection region:

$$
V = \int_{\theta_1}^{\theta_2} \int_{r=0}^{R(\theta)} \left(\text{top surface} - \text{bottom surface} \right) r \, dr \, d\theta
$$

Given the surfaces:
- Top surface: $z = r (\cos \theta + \sin \theta)$,
- Bottom surface: $z=0$,

The integral becomes:

$$
V = \int_{\theta_1}^{\theta_2} \int_{0}^{R(\theta)} r \times r (\cos \theta + \sin \theta) \, dr \, d\theta
$$
which simplifies to:

$$
V = \int_{\theta_1}^{\theta_2} \int_{0}^{R(\theta)} r^2 (\cos \theta + \sin \theta) \, dr \, d\theta
$$

Performing the inner integral:

$$
\int_{0}^{R(\theta)} r^2 \, dr = \frac{R(\theta)^3}{3}
$$

So,

$$
V = \frac{1}{3} \int_{\theta_1}^{\theta_2} R(\theta)^3 (\cos \theta + \sin \theta) \, d\theta
$$

The exact form of $R(\theta)$ depends on the boundary in the xy-plane.

Explicit Calculation for a Specific Boundary

Case: Circular Boundary at Radius R

Suppose the boundary is a circle of radius R, so $R(\theta) = R$.

The limits of θ are as previously determined:
$$
- \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}
$$

The volume becomes:

$$
V = \frac{R^3}{3} \int_{-\pi/4}^{3\pi/4} (\cos \theta + \sin \theta) \, d\theta
$$

Compute the integral:

\[
\int (\cos \theta + \sin \theta) \, d\theta = \sin \theta - \cos \theta + C
\]

Evaluate from \(-\pi/4 \) to \(3\pi/4 \):

\[
V = \frac{R^3}{3} \left[(\sin \theta - \cos \theta) \Big|_{-\pi/4}^{3\pi/4} \right]
\]

Calculating:

- At \(\theta = 3\pi/4 \):

\[
\sin \left(\frac{3\pi}{4} \right) = \frac{\sqrt{2}}{2}
\]
\[
\cos \left(\frac{3\pi}{4} \right) = - \frac{\sqrt{2}}{2}
\]

- At \(\theta = -\pi/4 \):

\[
\sin \left(-\frac{\pi}{4} \right) = - \frac{\sqrt{2}}{2}
\]
\[
\cos \left(-\frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}
\]

Thus,

\[
V = \frac{R^3}{3} \left[\left(\frac{\sqrt{2}}{2} - (-\frac{\sqrt{2}}{2}) \right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right)

Frequently Asked Questions

How do you set up the triple integral in cylindrical coordinates for the volume enclosed by the cone z = x + y?

Convert $x + y$ to cylindrical coordinates: $x = r \cos \theta$, $y = r \sin \theta$, so $x + y = r (\cos \theta + \sin \theta)$. The cone becomes $z = r (\cos \theta + \sin \theta)$. The limits are r from 0 to where the surface intersects the z-axis (often determined by the cone's bounds), θ from 0 to 2π, and z from 0 up to $r (\cos \theta + \sin \theta)$. The volume element is $dV = r\, dz\, dr\, d\theta$.

What is the significance of converting to cylindrical coordinates when finding the volume of this solid?

Cylindrical coordinates simplify the integration process by exploiting the symmetry of the cone around the z-axis. It transforms complex boundaries into more manageable limits, especially when the surface is expressed in terms of r and θ.

How do you determine the bounds for r and θ when computing the volume enclosed by the cone z = x + y?

θ ranges from 0 to 2π to cover the entire revolution around the z-axis. The bounds for r are determined by where the surface $z = r(\cos \theta + \sin \theta)$ intersects the base plane (z=0) or the top boundary if specified. Typically, r varies from 0 to the maximum radius where the surface intersects the plane or the boundary of the solid.

What is the formula for the volume of the solid enclosed by the cone in cylindrical coordinates?

The volume V is given by the triple integral $V = \iiint r \, dz \, dr \, d\theta$, with limits for z from 0 to $r(\cos \theta + \sin \theta)$, r from 0 to $R(\theta)$, and θ from 0 to 2π, where $R(\theta)$ is determined based on the intersection points.

How do you handle the case when the cone extends infinitely in the cylindrical coordinate system?

You need to identify the specific bounds where the solid is enclosed, typically by setting limits where the surface intersects other boundaries or by truncating at a certain height or radius. Without bounds, the volume would be infinite and not meaningful.

Can you explain the process of integrating to find the volume using cylindrical coordinates for this problem?

Yes. First, express the surface in cylindrical coordinates. Next, determine the limits for r and θ based on the geometry. Then, set up the triple integral with r as the Jacobian determinant (r), integrating z from 0 up to $r(\cos \theta + \sin \theta)$. Finally, perform the integrations step-by-step to find the volume.

Are there any symmetry properties that can simplify the calculation of the volume in this problem?

Yes. Since the cone involves symmetric functions of θ $(\cos \theta + \sin \theta)$, symmetry about certain axes can be used to simplify the integral, possibly reducing the limits or the integrand. Recognizing symmetry can help evaluate the integral more efficiently.

What challenges might arise when calculating the volume of this solid in cylindrical coordinates?

Challenges include accurately determining the bounds for r based on the intersection points, handling the angular dependence of the surface, and evaluating the resulting integrals, especially if the surface leads to complex limits or integrand expressions.

How does the equation z = x + y relate to the shape of the solid in cylindrical coordinates?

In cylindrical coordinates, $z = r (\cos \theta + \sin \theta)$. This expression describes a conical surface symmetric around the z-axis, with the shape depending on the angle θ. It defines the upper boundary of the solid in the volume calculation.

Additional Resources

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Introduction: Understanding the Problem and Its Significance

In the realm of multivariable calculus, the calculation of volumes bounded by surfaces is a fundamental task with wide-ranging applications—from physics and engineering to computer graphics and environmental modeling. Among the various coordinate systems available for such calculations, cylindrical coordinates stand out for their efficiency in handling problems exhibiting rotational symmetry around a central axis.

The problem at hand involves determining the volume of a solid enclosed by a specific cone defined by the equation $Z = X + Y$. Cones, as geometric entities, are intriguing due to their straightforward yet versatile structure, and their study often leads to insights into more complex three-dimensional shapes. When combined with the cylindrical coordinate system, the problem becomes more approachable, especially given the symmetry inherent in many conical surfaces.

This article provides a comprehensive exploration of how to leverage cylindrical coordinates to compute the volume of the solid enclosed by the cone $Z = X + Y$. We will delve into the geometric interpretation of the problem, establish the necessary bounds, and execute the integral calculation step-by-step. Along the way, the discussion will highlight the advantages of cylindrical coordinates and clarify the underlying mathematical principles.

---

1. Geometric Interpretation of the Cone $Z = X + Y$

Before proceeding with coordinate transformations, it is crucial to understand the geometric nature of the surface $Z = X + Y$.

1.1. Cartesian Representation

In Cartesian coordinates, the surface is given by:

$$
Z = X + Y
$$

This is a plane inclined at an angle to the XY-plane. Its intercepts are:

- When $X=0, Y=0$, then $Z=0$.
- When $X=1, Y=0$, then $Z=1$.
- When $X=0, Y=1$, then $Z=1$.

The plane cuts the XY-plane along the line $Z=0$, i.e., the XY-plane itself, and extends infinitely in the positive and negative directions depending on the domain restrictions.

1.2. Geometric Features

- Symmetry: The plane is symmetric with respect to the line $X=Y$, but not rotationally symmetric about a particular axis in Cartesian coordinates.
- Shape: Since the surface is a plane with a positive slope in both the X and Y directions, the volume enclosed will depend on the specific bounds in the XY-plane.

---

2. Transition to Cylindrical Coordinates

Cylindrical coordinates (r, θ, z) are defined by the transformations:

$$
\begin{cases}
x = r \cos \theta \\
y = r \sin \theta \\
z = z
\end{cases}
$$

The primary motivation for using cylindrical coordinates in this problem is their natural fit for surfaces involving X and Y, especially when the region exhibits rotational symmetry or can be simplified using radial variables.

2.1. Expressing the Surface $Z = X + Y$ in Cylindrical Coordinates

Substituting $X = r \cos \theta$ and $Y = r \sin \theta$, the surface becomes:

$$
Z = r \cos \theta + r \sin \theta = r (\cos \theta + \sin \theta)
$$

This simplifies the representation of the boundary surface in cylindrical coordinates.

---

3. Defining the Enclosed Volume

To find the volume of the solid enclosed by the cone $Z = X + Y$, we need to specify the domain in the XY-plane over which the volume extends.

3.1. Clarifying the Boundary Conditions

Typically, a problem involving the volume enclosed by a cone is specified with additional constraints, such as:

- The volume is bounded above by the surface $Z = X + Y$.
- The volume is bounded below by the plane $Z=0$.
- The boundary in the XY-plane is a specific region where the surface intersects.

Suppose the problem stipulates that the solid is enclosed by the cone $Z = X + Y$ and the plane $Z=0$, over a region in the XY-plane where $Z \geq 0$.

Given the surface $Z = r (\cos \theta + \sin \theta)$, the condition $Z \geq 0$ leads to:

$$
r (\cos \theta + \sin \theta) \geq 0
$$

which indicates that the domain of integration in the r and θ variables depends on the sign of $\cos \theta + \sin \theta$.

3.2. The Region in the XY-Plane

The projection of the volume onto the XY-plane is the set of points where:

$$
Z \geq 0 \quad \text{and} \quad Z \leq X + Y
$$

Since the volume is bounded below by the plane $Z=0$, and above by the surface $Z = X + Y$, the vertical bounds for Z at each point are:

$$
0 \leq Z \leq r (\cos \theta + \sin \theta)
$$

---

4. Determining the Region in Cylindrical Coordinates

To set up the integral, it is essential to identify the domain D in the $r-\theta$ plane.

4.1. Conditions for r and θ

The key boundary in the XY-plane is where the surface intersects the plane $Z=0$. This occurs when:

$$
r (\cos \theta + \sin \theta) = 0
$$

which implies:

$$
r=0 \quad \text{or} \quad \cos \theta + \sin \theta=0
$$

- At $r=0$: the origin is always part of the domain.
- When $\cos \theta + \sin \theta=0$: the boundary line in terms of θ.

Solving $\cos \theta + \sin \theta=0$:

$$
\sin \theta = - \cos \theta \Rightarrow \tan \theta = -1
$$

which yields:

$$
\theta = -\frac{\pi}{4} \quad \text{or} \quad \theta = \frac{3\pi}{4}
$$

Because $\tan \theta = -1$ at these angles, the region in θ where $\cos \theta + \sin \theta \geq 0$ is between $-\pi/4$ and $3\pi/4$.

---

5. Setting Up the Integral for the Volume

The volume V can be expressed as a double integral over the region D:

$$
V = \iint_D \left(\text{height} \right) \, dA
$$

where the height at each point is $Z = r (\cos \theta + \sin \theta)$, and the differential area element in cylindrical coordinates is:

$$
dA = r \, dr \, d\theta
$$

Thus,

\[
V = \int_{\theta = -\pi/4}^{3\pi/4} \int_{r=0}^{r_{max}(\theta)} r (\cos \theta + \sin \theta) \times r \, dr \, d\theta
\]

Note that for each \(\theta \), the maximum \(r \) is dictated by the condition \(Z \geq 0 \), which is satisfied for:

\[
r \geq 0 \quad \text{and} \quad r \leq R(\theta)
\]

where \(R(\theta) \) is the boundary in the \(r \)-direction.

Given that the surface \(Z = r (\cos \theta + \sin \theta) \) is unbounded unless restricted, we need an explicit boundary. For example, if the problem constrains the volume within a specific region such as a circle of radius \(R_0 \), or a bounded region in the \(XY \)-plane, the limits can be specified accordingly.

---

6. Calculating the Volume: Step-by-Step

Assuming the region is bounded such that:

\[
r \in [0, R(\theta)]
\]

and the boundary occurs where the surface intersects some upper bound in the \(XY \)-plane, or at a specific radius \(R_0 \).

Suppose, for simplicity, that the boundary in the \(XY \)-plane is a circle of radius \(R \), i.e., \(r \in [0, R] \), with \(R \) finite.

The volume integral becomes:

\[
V = \int_{\theta = -\pi/4}^{3\pi/4} \int_{r=0}^{R} r (\cos \theta + \sin \theta) \times r \, dr \, d\theta
\]

which simplifies to:

\[
V = \int_{-\pi/4}^{3\

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