Use Rectangular Coordinates To Calculate The Volume Of A Circular Paraboloid Described By Z= 4x^2 Y^2

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Calculating the volume of a three-dimensional shape like a circular paraboloid is a fundamental problem in multivariable calculus. When the paraboloid is described by the equation $Z = 4x^2 + y^2$, understanding how to approach the volume calculation using rectangular coordinates becomes essential. This article provides a comprehensive step-by-step guide to calculating the volume enclosed by such a paraboloid, employing rectangular (Cartesian) coordinates and multiple integral techniques.

Understanding the Geometric Shape: The Circular Paraboloid

Before diving into the calculations, it's important to understand the geometric nature of the paraboloid described by the equation:

$$
Z = 4x^2 + y^2
$$

This surface is symmetric about the z-axis and opens upward, forming a bowl-shaped surface that extends infinitely unless bounded by a specified region in the xy-plane.

Key Characteristics of the Paraboloid

- Vertex: The lowest point of the paraboloid, located at the origin $(0, 0, 0)$.
- Axis of symmetry: The z-axis.
- Cross-sections:
- Horizontal cross-sections (parallel to the xy-plane) are circles defined by $Z = \text{constant}$.
- Vertical cross-sections (parallel to the xz- or yz-plane) are parabolas.

Defining the Region for Volume Calculation

To calculate a finite volume, we need to specify the domain over which we are integrating. Usually, this involves choosing a boundary in the xy-plane, such as the projection of the paraboloid onto the xy-plane, up to a certain height Z_{max}, or bounded by a specific curve.

Choosing the Region in the xy-plane

Suppose we are asked to find the volume enclosed between the paraboloid and the xy-plane, up to a certain height $Z = Z_{max}$. Alternatively, if a specific boundary curve is given, we define the region R in the xy-plane.

For this guide, we will consider the volume enclosed between the paraboloid and the xy-plane, bounded by the surface $Z = 4x^2 + y^2$ over a region R.

Example: Volume enclosed up to a specific height

Let's consider the volume under the paraboloid between $Z=0$ and $Z=H$, where H is a positive constant.

From the equation $Z=4x^2 + y^2$, the projection onto the xy-plane corresponds to the set of points satisfying:

$$
4x^2 + y^2 \leq H
$$

This inequality describes an ellipse in the xy-plane.

Converting to Rectangular Coordinates: Setting Up the Integral

Using rectangular coordinates, the volume V can be expressed as a double integral over the region R:

$$
V = \iint_{R} Z \, dA
$$

Since $Z = 4x^2 + y^2$, the integral becomes:

$$
V = \iint_{R} (4x^2 + y^2) \, dx\,dy
$$

where R is the projection of the volume onto the xy-plane.

Defining the Integration Region R

Given the boundary $4x^2 + y^2 \leq H$, the region R is an ellipse centered at the origin with semi-axes:

- \(a = \frac{\sqrt{H}}{2} \) along the x-axis
- \(b = \sqrt{H} \) along the y-axis

Expressed explicitly:

\[
R = \left\{ (x, y) \mid 4x^2 + y^2 \leq H \right\}
\]

Setting Up the Double Integral in Rectangular Coordinates

The integral for volume becomes:

\[
V = \iint_{4x^2 + y^2 \leq H} (4x^2 + y^2) \, dx\,dy
\]

To evaluate this integral, it’s often convenient to perform a substitution that simplifies the elliptical region into a circular one, such as transforming to scaled coordinates.

Coordinate Transformation

Define new variables:

\[
u = 2x \quad \Rightarrow \quad x = \frac{u}{2}
\]
\[
v = y \quad \Rightarrow \quad y = v
\]

The Jacobian of this transformation is:

\[
J = \left| \frac{\partial (x, y)}{\partial (u, v)} \right| = \left|
\begin{matrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{matrix}
\right| =
\left|
\begin{matrix}
\frac{1}{2} & 0 \\

0 & 1
\end{matrix}
\right| = \frac{1}{2}
\]

The differential area element transforms as:

\[
dx\,dy = \left| J \right| du\,dv = \frac{1}{2} du\,dv
\]

The boundary condition becomes:

\[
4x^2 + y^2 = u^2 + v^2 \leq H
\]

which describes a circle of radius \(\sqrt{H} \) in the \(uv \)-plane.

Rewriting the Integral in Transformed Coordinates

The integral becomes:

\[
V = \iint_{u^2 + v^2 \leq H} \left[4 \left(\frac{u}{2} \right)^2 + v^2 \right] \, dx\,dy
\]

Expressed in terms of \(u \) and \(v \):

\[
V = \iint_{u^2 + v^2 \leq H} \left(u^2 + v^2 \right) \times \frac{1}{2} du\,dv
\]

Simplify:

\[
V = \frac{1}{2} \iint_{u^2 + v^2 \leq H} (u^2 + v^2) \, du\,dv
\]

Switching to Polar Coordinates for Simplification

In the \(uv \)-plane, the region is a circle of radius \(R = \sqrt{H} \), making polar coordinates ideal:

\[
u = r \cos \theta, \quad v = r \sin \theta
\]

$$
du\,dv = r dr d\theta
$$

The integral becomes:

$$
V = \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{\sqrt{H}} r^2 \times r dr d\theta = \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{\sqrt{H}} r^3 dr d\theta
$$

Evaluating the inner integral:

$$
\int_{0}^{\sqrt{H}} r^3 dr = \frac{1}{4} r^4 \bigg|_{0}^{\sqrt{H}} = \frac{1}{4} (\sqrt{H})^4 = \frac{1}{4} H^2
$$

Now, the outer integral:

$$
V = \frac{1}{2} \times 2\pi \times \frac{1}{4} H^2 = \frac{1}{2} \times 2\pi \times \frac{H^2}{4} = \frac{\pi H^2}{4}
$$

Final Expression for the Volume

The volume enclosed between the paraboloid $Z = 4x^2 + y^2$ and the xy-plane, up to height $Z = H$, is:

$$
\boxed{
V = \frac{\pi H^2}{4}
}
$$

This expression allows you to compute the volume for any given height H.

Special Case: Volume up to $Z=H$ with no upper boundary

If the paraboloid extends infinitely upward, the volume becomes unbounded. To have a finite volume, the region must be bounded by a specific height H, or by a closed boundary such as a circle or ellipse in the xy-plane.

Summary of the Calculation Steps

To summarize the process of calculating the volume of a paraboloid using rectangular coordinates:

1. Identify the surface equation: $Z = 4x^2 + y^2$.

2. Determine the bounded region R in the xy-plane corresponding to the volume of interest.

3. Express the volume as a double integral:

\[
V = \i

Frequently Asked Questions

How do I set up the integral to find the volume of the paraboloid defined by $z = 4x^2 + y^2$?

You convert to cylindrical coordinates or use rectangular coordinates with appropriate limits. Typically, you identify the projection of the paraboloid onto the xy-plane and integrate z over that region. For $z=4x^2 + y^2$, the volume can be found by integrating over the region where the paraboloid exists, often bounded by $z=0$ or a specified surface.

What is the projection region in the xy-plane for the paraboloid $z=4x^2 + y^2$?

The projection region depends on the limits of integration. If the paraboloid is bounded above by a plane or a specific z-value, then the projection is the set of (x, y) satisfying $4x^2 + y^2 \leq$ some constant. For example, if z is bounded by $z= R$, then $4x^2 + y^2 \leq R$.

How can I convert the integral from rectangular to polar coordinates when calculating volume?

Replace $x = r \cos \theta$ and $y = r \sin \theta$, and express z in terms of r and θ. The Jacobian determinant for polar coordinates is r, so the double integral becomes $\iint z(r, \theta)\, r\, dr\, d\theta$ over the appropriate limits based on the projection region.

What are the limits of integration in polar coordinates for the region underneath the paraboloid?

The limits for r and θ depend on the projection region. Usually, θ varies from 0 to 2π, and r varies from 0 to the boundary determined by the maximum radius where the paraboloid reaches the specified z-value or the boundary of the region.

How do I determine the volume when the paraboloid is bounded above by a plane z=H?

Set up the integral over the projection region where $4x^2 + y^2 \leq R$, with R determined by the condition $4x^2 + y^2 = H$. Convert to polar coordinates, with r from 0 to $\sqrt{H}$, and θ from 0 to 2π, then integrate $z = 4x^2 + y^2$ over this region.

Can I use cylindrical coordinates directly to compute the volume of the paraboloid?

Yes, converting to cylindrical coordinates (r, θ, z) simplifies the integration because of the symmetry. The paraboloid becomes $z = 4r^2 \cos^2 \theta + r^2 \sin^2 \theta$, which simplifies to $z = r^2(4 \cos^2 \theta + \sin^2 \theta)$. You can then integrate over r and θ accordingly.

What is the integral setup for calculating the volume of the paraboloid using rectangular coordinates?

Set up a double integral over the projection region in xy-plane: Volume = $\iint z \, dx \, dy$, where $z = 4x^2 + y^2$. The limits depend on the region chosen, such as a circle or bounded region, and integrate accordingly.

What are common mistakes to avoid when calculating volume using rectangular coordinates for this paraboloid?

Common mistakes include incorrect limits of integration, forgetting to include the Jacobian when switching to polar coordinates, and misidentifying the projection region. Always carefully determine the bounds based on the shape and bounds of the paraboloid.

How does symmetry help in calculating the volume of the paraboloid z=4x^2 + y^2?

Symmetry about the axes allows simplifying the integral by using polar coordinates, reducing the problem to a single integral over a radial variable and an angular variable, which makes the calculation more straightforward.

Additional Resources

Using Rectangular Coordinates to Calculate the Volume of a Circular Paraboloid Described by $Z = 4x^2 + y^2$

Calculating the volume beneath a surface is a fundamental problem in multivariable calculus, and understanding how to do so using different coordinate systems is crucial for tackling a wide range of mathematical and engineering problems. In this detailed review, we focus on the specific example of a circular paraboloid described by the equation $Z = 4x^2 + y^2$, illustrating the process of using rectangular coordinates (Cartesian coordinates) to evaluate its volume.

Introduction to the Problem

The surface in question is given by:

$$Z = 4x^2 + y^2$$

This describes a paraboloid opening upward, symmetric with respect to both the x- and y-axes, with its vertex at the origin (0,0,0). The goal is to compute the volume enclosed beneath this surface over a specified region in the xy-plane.

Typically, such problems involve:

- Defining the region of integration in the xy-plane.
- Setting up the double integral over that region.
- Evaluating the integral to find the volume.

In many instances, the symmetry of the surface suggests switching to circular (polar) coordinates for simplicity. However, here, we focus on using rectangular coordinates exclusively, which involves setting appropriate bounds and directly integrating in x and y.

Understanding the Geometry of the Paraboloid

Before setting up the integral, it’s essential to understand the shape of the paraboloid and the region over which we integrate.

Shape and Symmetry

- The paraboloid is symmetric with respect to both axes.
- Its shape resembles a bowl, with the lowest point at the vertex (origin).
- The "height" at a point (x, y) is given by $Z = 4x^2 + y^2$.

Intersections with Planes

To define a specific volume, we need to specify the boundary — usually a plane or a particular level surface.

Suppose the problem asks for the volume bounded above by a certain height, say $Z = h$. Then, the boundary surface becomes:

$$ 4x^2 + y^2 = h $$

which is an ellipse in the xy-plane.

Setting Up the Double Integral in Rectangular Coordinates

The general approach involves:

1. Identifying the boundary in the xy-plane.
2. Expressing the limits of integration for x and y.
3. Formulating the double integral for volume.

Choosing the Boundary: The Horizontal Plane $Z = h$

Let’s consider the volume bounded above by the paraboloid and below by the xy-plane. To compute the volume up to a certain height $Z = h$:

$$ 0 \leq Z \leq 4x^2 + y^2 $$
$$ \Rightarrow 0 \leq Z \leq h $$

The intersection of the paraboloid with the plane $Z=h$ occurs where:

$$ 4x^2 + y^2 = h $$

which describes an ellipse in the xy-plane.

Important note: If the problem specifies a specific boundary in xy (like a circle or an ellipse), we can adapt accordingly. For now, assume we want the volume bounded above by the paraboloid and below by the xy-plane, over the region where the paraboloid is less than or equal to h.

Expressing the Region of Integration

The region in the xy-plane under the paraboloid $Z=h$ is:

$$ 4x^2 + y^2 \leq h $$

which is an ellipse centered at the origin, with:

- Semi-major axis in x-direction: $a = \sqrt{\frac{h}{4}} = \frac{\sqrt{h}}{2}$
- Semi-minor axis in y-direction: $b = \sqrt{h}$

The limits for y at a fixed x are:

$$ y \in \left[-\sqrt{h - 4x^2}, \sqrt{h - 4x^2} \right] $$

and x varies over:

$$ x \in \left[-\frac{\sqrt{h}}{2}, \frac{\sqrt{h}}{2} \right] $$

Formulating the Volume Integral

The volume V beneath the paraboloid up to height h is:

$$
V = \iint_{D} Z \, dA
$$

where D is the elliptical region:

$$
V = \iint_{4x^2 + y^2 \leq h} (4x^2 + y^2) \, dx\,dy
$$

Expressed explicitly:

$$
V = \int_{x = -\frac{\sqrt{h}}{2}}^{\frac{\sqrt{h}}{2}} \int_{y = -\sqrt{h - 4x^2}}^{\sqrt{h - 4x^2}} (4x^2 + y^2) \, dy\, dx
$$

Calculating the Double Integral

The integral involves integrating a quadratic function over an elliptical region.

Step 1: Inner integral over y

For fixed x:

\[

I(x) = \int_{-\sqrt{h - 4x^2}}^{\sqrt{h - 4x^2}} (4x^2 + y^2) \, dy
\]

Because the integrand separates into terms involving \(y \):

\[
I(x) = 4x^2 \int_{-A(x)}^{A(x)} dy + \int_{-A(x)}^{A(x)} y^2 \, dy
\]

where:

\[
A(x) = \sqrt{h - 4x^2}
\]

Calculating each:

- \(\int_{-A}^{A} dy = 2A\)
- \(\int_{-A}^{A} y^2 dy = 2 \int_{0}^{A} y^2 dy = 2 \times \frac{A^3}{3} = \frac{2A^3}{3}\)

Thus:

\[
I(x) = 4x^2 \times 2A + \frac{2A^3}{3} = 8x^2 A + \frac{2A^3}{3}
\]

Substituting back \(A = \sqrt{h - 4x^2} \):

\[
I(x) = 8x^2 \sqrt{h - 4x^2} + \frac{2}{3} (h - 4x^2)^{3/2}
\]

Step 2: Outer integral over x

The volume becomes:

\[
V = \int_{-\frac{\sqrt{h}}{2}}^{\frac{\sqrt{h}}{2}} \left[8x^2 \sqrt{h - 4x^2} + \frac{2}{3} (h - 4x^2)^{3/2} \right] dx
\]

Because of symmetry, the integrand is an even function:

- \(8x^2 \sqrt{h - 4x^2} \) is even (since \(x^2 \) is even, and the root is even).
- \((h - 4x^2)^{3/2} \) is even.

Therefore, we can write:

$$
V = 2 \int_{0}^{\frac{\sqrt{h}}{2}} \left[8x^2 \sqrt{h - 4x^2} + \frac{2}{3} (h - 4x^2)^{3/2} \right] dx
$$

Evaluating the Integral: Change of Variables

To evaluate this integral explicitly, substitution simplifies the integrand.

Substitution: $u = 4x^2$

Let:

$$
u = 4x^2 \Rightarrow du = 8x dx
$$

When $x = 0$:

$$
u = 0
$$

When $x = \frac{\sqrt{h}}{2}$:

$$
u = 4 \times \left(\frac{\sqrt{h}}{2} \right)^2 = 4 \times \frac{h}{4} = h
$$

Express dx in terms of du:

$$
du = 8x dx \Rightarrow dx = \frac{du}{8x}
$$

But $x = \frac{\sqrt{u}}{2}$, so:

$$
dx = \frac{du}{8 \times \frac{\sqrt{u}}{2}} = \frac{du}{4 \sqrt{u}}
$$

Rewriting the integral:

$$
V = 2
$$

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