Use Stokes' Theorem To Evaluate F. Dr Where C Is Oriented Counterclockwise As Viewed From Above. F(x,

Use Stokes' Theorem To Evaluate F · Dr Where C Is Oriented Counterclockwise As Viewed From Above

Use Stokes' Theorem To Evaluate F · Dr Where C Is Oriented Counterclockwise As Viewed From Above is a fundamental problem in vector calculus, especially when dealing with line integrals and surface integrals. This theorem provides an elegant way to convert a line integral over a closed curve into a surface integral over the surface it encloses, simplifying complex calculations and offering deeper insights into the nature of vector fields. Whether you're a student learning about vector calculus or a professional applying these principles in physics or engineering, understanding how to use Stokes' Theorem effectively is an invaluable skill.

Understanding the Foundations of Stokes' Theorem

What Is Stokes' Theorem?

Stokes' Theorem relates a surface integral of the curl of a vector field to a line integral of the vector field itself. Mathematically, it is expressed as:

$$
\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}
$$

where:
- C is a positively oriented, smooth, closed curve,
- S is a surface bounded by C,
- $\mathbf{F}$ is a vector field,
- $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$.

This theorem is particularly useful when the curl of the vector field is easier to evaluate over a surface than directly computing the line integral.

Orientation of the Curve and Surface

The orientation of the curve C and the surface S are crucial:
- The curve C is oriented counterclockwise as viewed from above, which determines the direction of the normal vector $d\mathbf{S}$.
- The surface S should be oriented consistently with this direction, following the right-hand rule.

Proper understanding of orientation ensures the correct application of Stokes' Theorem and accurate calculations.

Step-by-Step Approach to Applying Stokes' Theorem

Applying Stokes' Theorem involves several systematic steps:

1. Identify the Vector Field $\mathbf{F}$

Begin by clearly defining the vector field involved in the problem. For example:

$$
\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}
$$

Ensure you understand each component and its domain.

2. Determine the Curve C and Its Orientation

- Identify the closed curve C over which the line integral is to be evaluated.
- Confirm that C is oriented counterclockwise when viewed from above.
- Visualize or sketch the curve and the surface S it bounds, maintaining consistent orientation.

3. Compute the Curl of $\mathbf{F}$

Calculate:

$$
\nabla \times \mathbf{F} =
$$

\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} &
\frac{\partial}{\partial z} \\
P & Q & R
\end{vmatrix}
\]

This step simplifies the problem to evaluating a surface integral of the curl.

4. Choose an Appropriate Surface \(S \)

Select a surface \(S \) bounded by \(C \) that makes the surface integral easier to evaluate. Common choices include:
- Flat planes when \(C \) is a simple closed curve.
- Surfaces with convenient parameterizations.

Ensure the surface's orientation matches the direction of \(C \) as per the right-hand rule.

5. Parameterize the Surface \(S \)

Express \(S \) using parameters \(u \) and \(v \):

\[
\mathbf{r}(u, v) = \textbf{function of } u, v
\]

Determine \(d\mathbf{S} \) as:

\[
d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) du dv
\]

where \(\mathbf{r}_u \) and \(\mathbf{r}_v \) are partial derivatives of \(\mathbf{r}(u, v) \).

6. Set Up and Evaluate the Surface Integral

Calculate:

\[
\iint_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}
\]

by substituting the parameterization and integrating over the domain in u, v.

7. Confirm the Orientation and Final Result

Verify that the orientation of S aligns with the direction of C. Adjust the sign of the integral if necessary.

Practical Example: Applying Stokes' Theorem

Let’s consider a practical example to illustrate the process:

Suppose $\mathbf{F}(x, y, z) = y \mathbf{i} + x \mathbf{j} + z \mathbf{k}$, and C is the circle $x^2 + y^2 = 1$ in the plane $z = 0$, oriented counterclockwise as viewed from above.

Step 1: Identify $\mathbf{F}$:

$$
\mathbf{F}(x, y, z) = y \mathbf{i} + x \mathbf{j} + z \mathbf{k}
$$

Step 2: Curve C:

- Circle of radius 1 in the xy-plane.
- Oriented counterclockwise from above.

Step 3: Compute curl:

$$
\nabla \times \mathbf{F} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial_x & \partial_y & \partial_z \\
y & x & z
\end{vmatrix} = (1 - 0)\mathbf{i} + (0 - 1)\mathbf{j} + (1 - 0)\mathbf{k} = \mathbf{i} - \mathbf{j} + \mathbf{k}
$$

Step 4: Choose surface S:

- The disk $x^2 + y^2 \leq 1$ in $z=0$, with upward normal.

Step 5: Parameterize S:

$$
\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} +

$$
0 \mathbf{k}
$$
with $r \in [0, 1]$, $\theta \in [0, 2\pi]$.

Compute:

$$
d\mathbf{S} = \mathbf{r}_r \times \mathbf{r}_\theta dr d\theta
$$

where:

$$
\mathbf{r}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}
$$
$$
\mathbf{r}_\theta = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j}
$$

The cross product:

$$
\mathbf{r}_r \times \mathbf{r}_\theta = r \mathbf{k}
$$

so:

$$
d\mathbf{S} = r dr d\theta \mathbf{k}
$$

Step 6: Evaluate surface integral:

$$
\iint_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{0}^{2\pi} \int_{0}^{1} (\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (r \mathbf{k}) dr d\theta
$$

Dot product:

$$
(\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot \mathbf{k} = 1
$$

Thus:

$$
\int_{0}^{2\pi} \int_{0}^{1} r \times 1 \ dr d\theta = \int_{0}^{2\pi} \left[\frac{r^2}{2}\right]_0^1 d\theta = \int_{0}^{2\pi} \frac{1}{2} d\theta = \pi
$$

\]

Final result: The value of the line integral \(\oint_{C} \mathbf{F} \cdot d\mathbf{r} \) is \(\pi \).

Advantages of Using Stokes' Theorem

- Simplifies complex calculations by converting line integrals into surface integrals.
- Provides geometric insight into the relationship between circulation and curl.
- Facilitates calculations in complicated geometries where direct line integrals are difficult.
- Useful in physics, especially in electromagnetism and fluid dynamics.

Common Mist

Frequently Asked Questions

What is Stokes' Theorem and how does it relate line integrals to surface integrals?

Stokes' Theorem states that the line integral of a vector field F around a closed curve C is equal to the surface integral of the curl of F over a surface S bounded by C, i.e., $\oint_C F \cdot dr = \iint_S (\text{curl } F) \cdot n \, dS$. It connects circulation around a curve to the curl of the field over the surface.

When applying Stokes' Theorem to evaluate the line

integral of F, what information about the curve C and surface S is essential?

It's essential to know the orientation of C (counterclockwise as viewed from above), the surface S bounded by C, and the vector field F (or its curl). The orientation of S must be consistent with C's orientation according to the right-hand rule.

How do you determine the orientation of the surface S when using Stokes' Theorem with a given curve C?

The orientation of S is chosen so that the boundary curve C, when traversed in its given direction (counterclockwise from above), has its positive normal vector pointing in the direction consistent with the right-hand rule. This ensures the surface integral correctly matches the line integral.

What are the steps to evaluate F · dr using Stokes' Theorem for a given vector field F and curve C?

First, identify the surface S bounded by C and find its normal vector. Next, compute the curl of F, then evaluate the surface integral of (curl F) · n over S. The resulting value equals the line integral of F · dr around C.

If F(x, y, z) is given and C is a circle in the xy-plane, how does the orientation affect the evaluation of the surface integral?

The orientation determines the direction of the normal vector used in the surface integral. For a counterclockwise traversal viewed from above, the normal vector points upward. Using this orientation ensures the surface integral aligns with the line integral according to Stokes' Theorem.

What are common mistakes to avoid when applying Stokes' Theorem to evaluate F?

Common mistakes include mismatching the orientation of C and the surface S, forgetting to compute the curl of F before integration, and neglecting the correct orientation conventions, which can lead to incorrect sign or value of the integral.

How does the choice of surface S affect the evaluation of the surface integral in Stokes' Theorem?

The surface S must be bounded by the given curve C and oriented consistently with C. Different surfaces bounded by the same curve

will yield the same integral value as long as the orientation is consistent, due to the independence of the line integral on the surface choice.

Additional Resources

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Stokes' Theorem is a fundamental concept in vector calculus that bridges the gap between line integrals and surface integrals, providing a powerful tool for evaluating circulation and flux in various physical and mathematical contexts. When faced with a vector field \(F\) and a closed curve \(C\), especially when the curve is oriented counterclockwise as viewed from above, Stokes' Theorem offers an elegant method to convert the line integral around \(C\) into a surface integral over the surface \(S\) bounded by \(C\). This approach simplifies complex calculations and enhances our understanding of the underlying vector fields.

In this comprehensive review, we will explore how to effectively apply Stokes' Theorem to evaluate the line integral \(\oint_C F \cdot dr \), where \(F\) is a given vector field, and \(C\) is an oriented curve. We will focus on the scenario where \(C\) is oriented counterclockwise from above, discuss the key steps involved in the process, analyze the features and nuances of the theorem, and provide illustrative examples to solidify understanding.

Understanding Stokes' Theorem: The Foundation for Surface and Line Integrals

Definition and Statement of Stokes' Theorem

Stokes' Theorem states that for a smooth surface \(S\) bounded by a simple, closed curve \(C\), the line integral of a vector field \(F\) around \(C\) is equal to the surface integral of the curl of \(F\) over \(S\):

\[
\oint_C F \cdot dr = \iint_S (\nabla \times F)

\cdot n \, dS
\]

where:

- \(F\) is a vector field \(F(x, y, z) = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}\).

- \(C\) is the boundary curve of \(S\), oriented according to the right-hand rule.

- \(n\) is the unit normal vector to \(S\), oriented according to \(C\)'s orientation.

This theorem essentially transforms a potentially complicated line integral into a more manageable surface integral involving the curl of \(F\).

Relevance in Physical and Mathematical Contexts

- Fluid Dynamics: Evaluating circulation around a loop to understand vortex behavior.

- Electromagnetism: Computing magnetic flux or circulation in field analysis.

- Mathematics: Simplifying calculations in vector calculus, especially when the curl is easier to compute than the line integral directly.

Preparing to Use Stokes' Theorem: Key Steps

Applying Stokes' Theorem involves a sequence of logical steps, ensuring accuracy and clarity in the calculation process.

1. Identify the Curve C and Surface S

- Curve C: The boundary of the surface S. Confirm the orientation (counterclockwise as viewed from above) aligns with the right-hand rule to determine the direction of the normal vector n.

- Surface S: Usually a planar region or a smooth surface bounded by C.

2. Determine the Orientation

Since the problem specifies that C is oriented counterclockwise from above, the normal vector n should point upward (positive z-direction) if the surface is planar and horizontal. Correct orientation is crucial because the sign of the surface integral depends on it.

3. Compute the Curl of F

Calculate $\nabla \times F$:

$$
\nabla \times F =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial_x & \partial_y & \partial_z \\
P & Q & R
\end{vmatrix}
= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\mathbf{i}
- \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right)\mathbf{j}
+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\mathbf{k}
$$

This step is often the most algebraically

intensive but is critical for the correct evaluation of the surface integral.

4. Parameterize the Surface S

Choose an appropriate parameterization $r(u, v)$ of S. For planar surfaces, this might be straightforward (e.g., $x = u, y = v, z = \text{constant}$). For curved surfaces, more sophisticated parameterizations are necessary.

5. Compute the Surface Integral

Express $(\nabla \times F) \cdot n \, dS$ in terms of the parameters, and evaluate the double integral over the domain D in (u, v)-space.

Applying the Method: Step-by-Step Example

To clarify the process, consider a concrete example:

Example: Let \(F(x, y, z) = (y, -x, 0)\), and \(C\) be the circle \(x^2 + y^2 = 1\) in the plane \(z=0\). The curve \(C\) is oriented counterclockwise as viewed from above.

Step 1: Identify \(C\) and \(S\)

- \(C\): The unit circle in the \(xy\)-plane, oriented counterclockwise.

- \(S\): The disk \(x^2 + y^2 \leq 1\), \(z=0\).

Step 2: Determine the orientation

- Since \(C\) is counterclockwise from above, the normal vector \(n\) points upward, i.e., \(n = \mathbf{k}\).

Step 3: Compute \(\nabla \times F\)

\[
\nabla \times F =
\begin{vmatrix}

\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial_x & \partial_y & \partial_z \\
y & -x & 0
\end{vmatrix}
= \left(\frac{\partial 0}{\partial y} -
\frac{\partial (-x)}{\partial z}
\right)\mathbf{i} - \left(\frac{\partial
0}{\partial x} - \frac{\partial y}{\partial z}
\right)\mathbf{j} + \left(\frac{\partial (-
x)}{\partial x} - \frac{\partial y}{\partial y}
\right)\mathbf{k}
\]

Calculating derivatives:

- \(\frac{\partial 0}{\partial y} = 0\)

- \(\frac{\partial (-x)}{\partial z} = 0\)

- \(\frac{\partial 0}{\partial x} = 0\)

- \(\frac{\partial y}{\partial z} = 0\)

- \(\frac{\partial (-x)}{\partial x} = -1\)

- \(\frac{\partial y}{\partial y} = 1\)

So,

\[

\nabla \times F = (0 - 0)\mathbf{i} - (0 - 0)\mathbf{j} + (-1 - 1)\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} - 2\mathbf{k}
\]

\[
\nabla \times F = -2 \mathbf{k}
\]

Step 4: Parameterize \(S\)

Use polar coordinates:

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z=0, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi
\]

The surface element:

\[
dS = r dr d\theta \, \mathbf{k}
\]

since the surface is the disk in the \(xy\)-plane with normal vector \(\mathbf{k}\).

Step 5: Compute the surface integral

$$
\iint_S (\nabla \times F) \cdot n \, dS = \iint_S (-2 \mathbf{k}) \cdot \mathbf{k} \, dS = \iint_S -2 \, dS
$$

Expressed in polar coordinates:

$$
\int_0^{2\pi} \int_0^1 -2 \, r dr d\theta
$$

Calculate:

$$
-2 \int_0^{2\pi} d\theta \int_0^1 r dr = -2 \cdot (2\pi) \cdot \left[\frac{r^2}{2} \right]_0^1 = -2 \cdot 2\pi \cdot \frac{1}{2} =
$$

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