Use The Annihilator Method To Find The General Solution Of The Differential Equation Y" - 2y 3y = E-!

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When faced with a second-order linear differential equation, especially one involving nonhomogeneous terms like exponential functions, the Annihilator Method offers a powerful approach to find the general solution. This method combines the techniques of solving the homogeneous part of the differential equation and applying an appropriate annihilator to manage the nonhomogeneous term effectively. In this comprehensive guide, we'll explore how to utilize the Annihilator Method to solve the differential equation:

$$ Y'' - 2Y' + 3Y = e^{-x} $$

by breaking down each step, understanding the underlying principles, and providing practical examples.

Understanding the Structure of the Differential Equation

Before delving into the Annihilator Method, it’s essential to comprehend the components of the differential equation:

- Type: Second-order linear differential equation
- Homogeneous part: $ Y'' - 2Y' + 3Y = 0 $
- Nonhomogeneous part: $ e^{-x} $

The goal is to find the general solution $ Y(x) $, which combines:

- The complementary (homogeneous) solution $ Y_c $
- The particular solution $ Y_p $

The Annihilator Method simplifies finding $ Y_p $ by transforming the nonhomogeneous term into a solution of a differential operator, enabling us to solve the entire equation systematically.

Step 1: Find the Homogeneous Solution

Begin by solving the homogeneous differential equation:

$$ Y'' - 2Y' + 3Y = 0 $$

Characteristic Equation:

$$
r^2 - 2r + 3 = 0
$$

Solve for r:

$$
r = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 3}}{2} = \frac{2 \pm \sqrt{4 - 12}}{2} = \frac{2 \pm \sqrt{-8}}{2}
$$

$$
r = 1 \pm i \sqrt{2}
$$

Homogeneous Solution:

$$
Y_c = e^{x} \left(C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x) \right)
$$

where C_1 and C_2 are arbitrary constants.

Summary:

- The homogeneous solution accounts for the complementary part of the general solution.
- These solutions typically involve exponential and trigonometric functions when roots are complex.

Step 2: Identify the Nonhomogeneous Term and Its Nature

The nonhomogeneous term is:

$$ e^{-x} $$

This exponential function suggests that the particular solution should include an exponential term, possibly multiplied by polynomial functions if needed.

Step 3: Find the Appropriate Annihilator

The core idea of the Annihilator Method is to find a differential operator that, when applied to the nonhomogeneous term, results in zero. This operator is called the annihilator.

Identify the annihilator for e^{-x}:

- The exponential function e^{ax} is annihilated by the differential operator $D - a$, where $D = \frac{d}{dx}$.

- For e^{-x}, the annihilator is:

$$
D + 1
$$

because:

$$
(D + 1) e^{-x} = \frac{d}{dx} e^{-x} + e^{-x} = - e^{-x} + e^{-x} = 0
$$

Construct the combined differential operator:

- To transform the original differential equation into an equation with zero on the right side, apply the annihilator:

$$
(D + 1)(Y'' - 2Y' + 3Y) = 0
$$

- This produces a higher-order homogeneous differential equation.

Step 4: Formulate the Annihilated Differential Equation

Applying the operator:

$$
(D + 1)(Y'' - 2Y' + 3Y) = 0
$$

gives:

\[

(D + 1)(Y'') - 2(D + 1)Y' + 3(D + 1)Y = 0
\]

which expands to:

\[
Y''' + Y'' - 2Y' - 2Y + 3Y' + 3Y = 0
\]

Simplify:

\[
Y''' + Y'' + (- 2Y' + 3Y') + (- 2Y + 3Y) = 0
\]

\[
Y''' + Y'' + Y' + Y = 0
\]

This is the annihilated differential equation.

Note:

- Since this is a third-order homogeneous differential equation, its general solution will include three arbitrary constants.

Step 5: Solve the Homogeneous Equation Derived from the Annihilator

Solve:

\[
Y''' + Y'' + Y' + Y = 0
\]

Characteristic Equation:

\[
r^3 + r^2 + r + 1 = 0
\]

Factor or find roots:

- Try rational root theorem: possible roots are \(\pm 1 \).

Test \(r = -1 \):

$$
(-1)^3 + (-1)^2 + (-1) + 1 = -1 + 1 - 1 + 1 = 0
$$

Yes! So, $r = -1$ is a root.

Factor out $r + 1$:

$$
r^3 + r^2 + r + 1 = (r + 1)(r^2 + 0r + 1) = (r + 1)(r^2 + 1)
$$

Solve:

$$
r^2 + 1 = 0 \Rightarrow r = \pm i
$$

Eigenvalues:

$$
r = -1, \quad r = i, \quad r = -i
$$

General Solution of the Annihilated Equation:

$$
Y_h = C_1 e^{-x} + C_2 \cos x + C_3 \sin x
$$

Step 6: Find the Particular Solution Y_p

Recall that the original nonhomogeneous term is e^{-x}. The homogeneous solution contains e^{-x} as well; hence, the particular solution must be modified accordingly to account for duplication (resonance).

Method:

- Since e^{-x} appears in Y_c, multiply by x to find a particular solution of the form:

$$
Y_p = A x e^{-x}
$$

- Alternatively, since the annihilator led to solutions involving e^{-x}, the particular solution should be of the form:

$$
Y_p = A x e^{-x}
$$

Determine A:

- Substitute Y_p into the original differential equation to solve for A.

Step 7: Compute Derivatives of Y_p and Solve for A

Let:

$$
Y_p = A x e^{-x}
$$

Compute derivatives:

$$
Y_p' = A e^{-x} - A x e^{-x} = A e^{-x} (1 - x)
$$

$$
Y_p'' = -A e^{-x} + A e^{-x} (1 - x) = -A e^{-x} + A e^{-x} - A x e^{-x} = -A x e^{-x}
$$

Now, substitute into the original differential equation:

$$
Y'' - 2Y' + 3Y = e^{-x}
$$

Plugging in:

$$
(-A x e^{-x}) - 2 [A e^{-x} (1 - x)] + 3 [A x e^{-x}] = e^{-x}
$$

Simplify:

$$
-A x e^{-x} - 2A e^{-x} (1 - x) + 3A x e^{-x} = e^{-x}
$$

$$

-A x e^{-x} - 2A e^{-x} + 2A x e^{-x} + 3A x e^{-x} = e^{-x}
\]

Combine like terms:

\[
(-A x + 2A x + 3A x) e^{-x} - 2A e^{-x} = e^{-x}
\]

\[
((-A + 2A + 3A) x - 2A) e^{-x} = e^{-x}
\]

\[
((4A) x - 2A) e^{-x} = e^{-x}
\]

Divide both sides by \(e^{-x} \):

\[
4A x - 2A = 1
\]

Since this must hold for all \(x \), equate coefficients:

- For the \(x \

Frequently Asked Questions

What is the Annihilator Method in solving differential equations?

The Annihilator Method involves finding an operator that, when applied to the nonhomogeneous term of a differential equation, results in zero. This helps in transforming the equation into a homogeneous form, allowing for easier solution finding.

How do you identify the form of the complementary (homogeneous) solution for Y'' - 2Y' + 3Y = 0?

Solve the characteristic equation associated with the homogeneous part, r^2 - 2r + 3 = 0, to find roots and determine the complementary solution based on those roots.

What is the nonhomogeneous part in the differential equation Y'' - 2Y' + 3Y = e^{-1}?

The nonhomogeneous term is e^{-1}, which is a constant multiple of the exponential function, representing the particular forcing term.

How do you find the annihilator for the forcing function e^{-1}?

Since e^{-1} is a constant exponential function, its annihilator is a differential operator like (D), because D applied to e^{ax} results in a constant multiple, and further application can annihilate constants.

What steps are involved in applying the Annihilator Method to this differential equation?

First, identify the annihilator for the forcing term, then multiply the original differential equation by this annihilator operator to obtain a higher-order homogeneous equation. Next, solve the homogeneous equation to find the general solution.

How do you determine the particular solution after applying the annihilator?

Once the equation is transformed into a homogeneous form, the particular solution corresponds to the part of the solution associated with the nonhomogeneous term, often found by solving the auxiliary equation or using the method of undetermined coefficients.

What is the importance of the roots of the characteristic equation in the solution process?

The roots determine the form of the complementary (homogeneous) solution. Real distinct roots lead to exponential solutions, repeated roots lead to polynomial multiplied solutions, and complex roots lead to oscillatory solutions involving sine and cosine.

Can the Annihilator Method be used for all types of differential equations?

While powerful, the Annihilator Method is mainly applicable to linear differential equations with constant coefficients and certain types of forcing functions. It may not be suitable for nonlinear or variable coefficient equations.

What is the final step after finding the homogeneous and particular solutions when using the Annihilator Method?

Combine the homogeneous solution with the particular solution to form the general solution of the differential equation, which encompasses all possible solutions.

Additional Resources

Annihilator Method: Unlocking the General Solution of Differential Equations

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Introduction

When delving into the realm of differential equations, particularly those with constant coefficients, the challenge often lies in efficiently finding the general solution that encapsulates all possible solutions. Among the arsenal of techniques, the Annihilator Method stands out as a powerful, systematic approach especially suitable for linear differential equations with non-homogeneous terms involving exponential, polynomial, or trigonometric functions.

In this article, we will embark on an in-depth exploration of the Annihilator Method, illustrating its application through a detailed example involving the differential equation:

\[Y'' - 2Y' + 3Y = e^{-t} \]

(Note: The original equation in your prompt appears to have typographical errors—assuming it to be a second-order linear differential equation with non-homogeneous exponential right-hand side, as this aligns with common uses of the Annihilator Method. If your specific equation differs, please clarify. For now, we proceed with this standard form.)

Understanding the Annihilator Method

What Is the Annihilator Method?

The Annihilator Method is a technique designed to find particular solutions to non-homogeneous linear differential equations with constant coefficients. It hinges on the idea of constructing an operator—called an annihilator—that, when applied to the non-homogeneous part of the differential equation, reduces it to zero. By applying this operator to the entire differential equation, the non-homogeneous term is eliminated, transforming the problem into a homogeneous differential equation, which can then be solved systematically.

Why Use the Annihilator Method?

- Efficiency: It provides a straightforward, algebraic way to find particular solutions without guessing.
- Systematic Approach: Especially effective for functions like exponentials, polynomials, sines, and cosines.
- Applicability: Suitable for linear differential equations with constant coefficients and specific types of non-homogeneous functions.

Step-by-Step Guide to Applying the Annihilator Method

Let’s walk through the process, breaking it down into manageable stages:

1. Identify the Differential Equation and Its Non-Homogeneous Part

Given:

$$ Y'' - 2Y' + 3Y = e^{-t} $$

- Homogeneous part: $ Y'' - 2Y' + 3Y = 0 $
- Non-homogeneous part: $ e^{-t} $

2. Determine the Complementary Solution

Solve the homogeneous equation:

$$ Y'' - 2Y' + 3Y = 0 $$

Characteristic equation:

$$ r^2 - 2r + 3 = 0 $$

Solve for $ r $:

$$ r = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 3}}{2} = \frac{2 \pm \sqrt{4 - 12}}{2} = \frac{2 \pm \sqrt{-8}}{2} $$

$$ r = 1 \pm i \sqrt{2} $$

Thus, the complementary (homogeneous) solution:

$$ Y_c(t) = e^{t} \left(C_1 \cos(\sqrt{2} t) + C_2 \sin(\sqrt{2} t) \right) $$

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3. Construct the Annihilator for the Non-Homogeneous Term

The non-homogeneous term is $ e^{-t} $. To find its annihilator:

- Recognize that $ e^{\alpha t} $ is annihilated by the differential operator:

$$ (D - \alpha) $$

where $ D = \frac{d}{dt} $.

- Since the non-homogeneous term is $ e^{-t} $, the annihilator is:

$$ (D + 1) $$

- For functions like $ e^{\alpha t} $, this operator annihilates the term:

$$ (D + 1) e^{-t} = 0 $$

---

4. Apply the Annihilator to the Entire Differential Equation

The original differential equation:

$$(D^2 - 2D + 3)Y = e^{-t}$$

Apply the annihilator $(D + 1)$:

$$(D + 1)(Y'' - 2Y' + 3Y) = (D + 1) e^{-t} = 0$$

This yields a higher-order homogeneous differential equation:

$$(D + 1)(D^2 - 2D + 3) Y = 0$$

Expanding:

$$(D + 1)(D^2 - 2D + 3)Y = 0$$

First, compute:

$$(D + 1)(D^2 - 2D + 3) = D(D^2 - 2D + 3) + (D^2 - 2D + 3)$$

which simplifies to:

$$D^3 - 2D^2 + 3D + D^2 - 2D + 3$$

Combine like terms:

$$D^3 - D^2 + D + 3$$

(Ensure proper distribution: actually, better to write explicitly)

Let me do the expansion carefully:

$$(D + 1)(D^2 - 2D + 3) = D \times (D^2 - 2D + 3) + 1 \times (D^2 - 2D + 3)$$

$$= D^3 - 2D^2 + 3D + D^2 - 2D + 3$$

Now combine like terms:

$$D^3 + (-2D^2 + D^2) + (3D - 2D) + 3$$

$$= D^3 - D^2 + D + 3$$

Therefore, the annihilated equation is:

$$(D^3 - D^2 + D + 3)Y = 0$$

---

5. Solve the Homogeneous Equation of the Higher Order

Now, find the roots of the characteristic equation:

$$ r^3 - r^2 + r + 3 = 0 $$

This cubic can be factored or solved via methods like Rational Root Theorem:

Possible rational roots:

$$ \pm 1, \pm 3 $$

Test $r=1$:

$$ 1 - 1 + 1 + 3 = 4 \neq 0 $$

Test $r=-1$:

$$ -1 - 1 - 1 + 3 = 0 $$

Yes! So, $r = -1$ is a root.

Factor out $(r + 1)$:

Divide the cubic by $(r+1)$:

Perform polynomial division or synthetic division:

Synthetic division:

Coefficients: 1 | -1 | 1 | 3

Bring down 1:

- Multiply: 1 (-1) = -1, add: -1 + (-1) = -2

- Multiply: -2 (-1) = 2, add: 1 + 2 = 3

- Multiply: 3 (-1) = -3, add: 3 + (-3) = 0

Remainder zero, so division is exact.

Quotient polynomial:

$$ r^2 - 2r + 3 $$

Set this quadratic equal to zero:

$$ r^2 - 2r + 3 = 0 $$

Discriminant:

$$ \Delta = (-2)^2 - 4 \times 1 \times 3 = 4 - 12 = -8 $$

Roots:

$$ r = \frac{2 \pm i \sqrt{8}}{2} = 1 \pm i \sqrt{2} $$

Roots:

- $r = -1$

- $r = 1 + i \sqrt{2}$

- $r = 1 - i \sqrt{2}$

---

6. Write the General Solution to the Higher-Order Homogeneous Equation

Corresponding to roots, the general solution:

$$
Y_p(t) = C_1 e^{-t} + e^{t} \left(C_2 \cos(\sqrt{2} t) + C_3 \sin(\sqrt{2} t) \right)
$$

---

7. Extract the Particular Solution

Recall that the particular solution corresponds to the part of the solution not contained within the complementary solution.

- The roots $r = -1$ and $r = 1 \pm i \sqrt{2}$ are distinct from the roots of the homogeneous equation $r = 1 \pm i \sqrt{2}$, which are associated with the original homogeneous solution.

- Crucially, because the annihilator $(D + 1)$ introduced the root $r = -1$, the particular solution will involve a term that accounts for this.

- The general form of the particular solution:

$$
Y_p(t) = A e^{-t}
$$

- However, since e^{-t} is a solution to the original homogeneous equation (as $r = -1$ is a root), we need to multiply by t to account for the duplication

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