Use The Laplace Transform To Solve The Given Initial-value Problem. Y'' + 4y = Sin T Scripted Capital

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When tackling differential equations, especially those involving initial conditions, the Laplace transform emerges as a powerful and efficient method. In this article, we will explore how to apply the Laplace transform to solve the initial-value problem defined by the differential equation $y'' + 4y = \sin T$, considering the initial conditions. We will break down each step, analyze the underlying principles, and demonstrate the solution process comprehensively to ensure clarity and understanding. Whether you're a student preparing for exams or a professional revisiting the fundamentals, this guide aims to deepen your grasp of using Laplace transforms in solving differential equations.

Understanding the Initial-Value Problem

Before diving into the solution, it is crucial to understand the problem's structure and the significance of the initial conditions.

Problem Statement

Given the second-order linear differential equation:

$$y'' + 4y = \sin T$$

with initial conditions:

$$y(0) = y_0, \quad y'(0) = y_1$$

where y_0 and y_1 are known constants.

Key Features of the Problem

- The differential equation is linear and homogeneous except for the right-hand side, which is a sinusoidal function.
- The initial conditions specify the state of the system at $T = 0$.
- The goal is to find $y(t)$, the solution function, satisfying both the differential equation and initial conditions.

Introduction to the Laplace Transform

The Laplace transform is a mathematical technique that transforms differential equations in the time domain into algebraic equations in the complex frequency domain. This transformation simplifies solving linear differential equations, especially with initial conditions.

Definition of the Laplace Transform

The Laplace transform $\mathcal{L}\{f(t)\}$ of a function $f(t)$, defined for $t \geq 0$, is:

$$F(s) = \int_0^{\infty} e^{-st}f(t) \, dt$$

where:

- s is a complex variable $s = \sigma + i \omega$,
- $F(s)$ is the transformed function.

Key Properties Relevant to Differential Equations

- Linearity: $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$
- Derivative: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
- Second derivative: $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
- Transform of sinusoidal functions: $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$

Applying Laplace Transform to the Differential Equation

To solve $y'' + 4y = \sin T$ using the Laplace transform, follow these systematic steps:

Step 1: Take the Laplace Transform of Both Sides

Applying $\mathcal{L}$ to the entire differential equation:

$$\mathcal{L}\{ y'' \} + 4 \mathcal{L}\{ y \} = \mathcal{L}\{ \sin T \}$$

Using the properties:

$$s^2Y(s) - s y(0) - y'(0) + 4Y(s) = \frac{1}{s^2 + 1}$$

where:

- $Y(s) = \mathcal{L}\{ y(t) \}$,
- $y(0) = y_0$,
- $y'(0) = y_1$,
- $\mathcal{L}\{ \sin T \} = \frac{1}{s^2 + 1}$.

Step 2: Rearrange the Equation for $Y(s)$

Combine like terms:
$$ (s^2 + 4)Y(s) - s y_0 - y_1 = \frac{1}{s^2 + 1} $$

Solve for $Y(s)$:
$$ Y(s) = \frac{1}{s^2 + 4} \left[\frac{1}{s^2 + 1} + s y_0 + y_1 \right] $$

This algebraic expression encapsulates the transformed solution.

Solving for $y(t)$ via Inverse Laplace Transform

The next step involves decomposing $Y(s)$ into simpler fractions, then applying the inverse Laplace transform to find $y(t)$.

Step 1: Express $Y(s)$ as a Sum of Simpler Terms

Rewrite:
$$ Y(s) = \frac{s y_0 + y_1}{s^2 + 4} + \frac{1}{(s^2 + 4)(s^2 + 1)} $$

This separation allows us to handle each term individually.

Step 2: Find the Inverse Laplace Transform of Each Term

- For $\frac{s y_0 + y_1}{s^2 + 4}$:

Recall:
$$ \mathcal{L}^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos a t $$
$$ \mathcal{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{\sin a t}{a} $$

Thus, depending on the constants y_0 and y_1, this term corresponds to a combination of sine and cosine functions.

- For $\frac{1}{(s^2 + 4)(s^2 + 1)}$:

Apply partial fraction decomposition to write:
$$ \frac{1}{(s^2 + 4)(s^2 + 1)} = \frac{A}{s^2 + 4} + \frac{B}{s^2 + 1} $$

Solve for A and B, then take inverse transforms accordingly.

Partial Fraction Decomposition and Inverse Transforms

Let's perform the partial fractions:

$$ \frac{1}{(s^2 + 4)(s^2 + 1)} = \frac{A}{s^2 + 4} + \frac{B}{s^2 + 1} $$

Multiply both sides by $ (s^2 + 4)(s^2 + 1) $:

$$ 1 = A(s^2 + 1) + B(s^2 + 4) $$

Set up equations:

$$ 1 = A s^2 + A + B s^2 + 4 B $$
$$ 1 = (A + B) s^2 + (A + 4 B) $$

Since this holds for all $ s $, equate coefficients:

1. Coefficient of $ s^2 $:

$$ A + B = 0 \Rightarrow B = -A $$

2. Constant term:

$$ A + 4 B = 1 $$

Substitute $ B = -A $:

$$ A + 4(-A) = 1 $$
$$ A - 4A = 1 $$
$$ -3A = 1 \Rightarrow A = -\frac{1}{3} $$

Then,

$$ B = -A = \frac{1}{3} $$

Now, rewrite $ Y(s) $:

$$ Y(s) = (s y_0 + y_1) \frac{1}{s^2 + 4} - \frac{1}{3} \frac{1}{s^2 + 4} + \frac{1}{3} \frac{1}{s^2 + 1} $$

The inverse Laplace transforms are well-known:

- $ \mathcal{L}^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos a t $
- $ \mathcal{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{\sin a t}{a} $

Thus, the solution $ y(t) $ is:

\[
\begin{aligned}
y(t) &= y_0 \cos 2 t + \frac{y_1}{2} \sin 2 t - \frac{1}{3} \cos 2 t + \frac{1}{3} \sin t \\

&= \left(y_0 - \frac{1}{3} \right) \cos 2 t + \frac{y_1}{2} \sin 2 t + \frac{1}{3} \sin t
\end{aligned}
\]

Complete Solution Summary

The explicit solution

Frequently Asked Questions

What is the first step in applying the Laplace transform to solve the initial-value problem y'' + 4y = Sin T?

The first step is to take the Laplace transform of both sides of the differential equation, using the known transforms for y'' and Sin T, while incorporating initial conditions if given.

How do you handle the Laplace transform of the second derivative y'' in solving the differential equation?

The Laplace transform of y'' is s^2 Y(s) - s y(0) - y'(0), where y(0) and y'(0) are initial conditions. This transforms the differential equation into an algebraic equation in terms of Y(s).

What is the Laplace transform of Sin T, and how is it used in solving the problem?

The Laplace transform of Sin T is 1 / (s^2 + 1). It replaces Sin T in the transformed equation, allowing us to solve for Y(s) algebraically.

How do initial conditions influence the solution when using Laplace transforms for differential equations?

Initial conditions determine the values of y(0) and y'(0), which appear in the transformed algebraic equation and influence the form of the solution Y(s), ultimately affecting the inverse Laplace transform step.

Once Y(s) is found, how do you obtain the solution y(t)?

You perform the inverse Laplace transform on Y(s) to find y(t). This may involve partial fraction decomposition and using known inverse transforms of standard functions.

What are common challenges when solving differential equations with Laplace transforms, and how can they be addressed?

Common challenges include partial fraction decomposition and handling initial conditions. These can be addressed by carefully simplifying Y(s), correctly applying initial conditions, and using standard inverse transform tables or methods.

Additional Resources

Use The Laplace Transform To Solve The Given Initial-value Problem: Y'' + 4y = Sin T

Introduction

The Laplace transform stands as one of the most powerful tools in the mathematician’s arsenal for solving linear differential equations, especially those accompanied by initial conditions. Its ability to convert differential equations into algebraic equations simplifies complex problems, making the path to solutions more straightforward. This article explores the application of the Laplace transform to a specific second-order initial-value problem:

$$ Y'' + 4y = \sin t $$

with initial conditions $y(0) = y_0$ and $y'(0) = y_1$ (assuming specific initial values). We will not only solve this particular differential equation but also delve into the theoretical underpinnings, the step-by-step process, and the broader implications of employing Laplace transforms in differential equations.

Background: The Laplace Transform and Its Relevance

What Is the Laplace Transform?

The Laplace transform is an integral transform that converts a function of a real variable t (usually time) into a function of a complex variable s. For a function $f(t)$, defined for $t \geq 0$, the Laplace transform $\mathcal{L}\{f(t)\}$ is given by:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt
$$

This transformation offers several advantages:

- Converts differential equations into algebraic equations.
- Simplifies handling initial conditions.
- Facilitates the use of tables of transforms for common functions.
- Provides a straightforward way to incorporate forcing functions like $\sin t$.

Why Use the Laplace Transform for the Given Problem?

The differential equation:

$$
Y'' + 4y = \sin t
$$

is a nonhomogeneous linear second-order differential equation with constant coefficients. Solving it directly involves finding the homogeneous solution and particular solution. The Laplace transform streamlines this process by turning derivatives into algebraic terms, leading to an easier algebraic manipulation and solution.

Theoretical Framework

Differential Equation in Standard Form

Before applying the Laplace transform, the differential equation needs to be expressed explicitly:

$$
\frac{d^2 y}{dt^2} + 4 y = \sin t
$$

with initial conditions:

$$
y(0) = y_0, \quad y'(0) = y_1
$$

Laplace Transform of Derivatives

Key properties of the Laplace transform for derivatives are:

$$
\mathcal{L}\{ y'(t) \} = sY(s) - y(0)
$$

$$
\mathcal{L}\{ y''(t) \} = s^2 Y(s) - s y(0) - y'(0)
$$

where $Y(s) = \mathcal{L}\{ y(t) \}$.

Step-by-Step Solution

Step 1: Apply the Laplace Transform to the Entire Differential Equation

Transform each term:

$$
\mathcal{L}\{ y'' \} + 4 \mathcal{L}\{ y \} = \mathcal{L}\{ \sin t \}
$$

Using the properties:

$$
s^2 Y(s) - s y_0 - y_1 + 4 Y(s) = \frac{1}{s^2 + 1}
$$

Step 2: Algebraic Manipulation

Group $Y(s)$ terms:

$$
(s^2 + 4) Y(s) = s y_0 + y_1 + \frac{1}{s^2 + 1}
$$

Solve for $Y(s)$:

$$
Y(s) = \frac{s y_0 + y_1}{s^2 + 4} + \frac{1}{(s^2 + 4)(s^2 + 1)}
$$

This expression decomposes into two parts:

- The homogeneous solution component
- The particular solution component

Step 3: Inverse Laplace Transform

To find $y(t)$, apply the inverse Laplace transform to each term separately.

Part A: Homogeneous component

$$
\mathcal{L}^{-1}\left\{ \frac{s y_0 + y_1}{s^2 + 4} \right\}
$$

Recognizing that:

$$
\mathcal{L}^{-1}\left\{ \frac{s}{s^2 + a^2} \right\} = \cos a t
$$

$$
\mathcal{L}^{-1}\left\{ \frac{1}{s^2 + a^2} \right\} = \frac{\sin a t}{a}
$$

Thus:

$$
y_h(t) = y_0 \cos 2 t + \frac{y_1}{2} \sin 2 t
$$

Part B: Particular component

Focus on:

$$
\mathcal{L}^{-1}\left\{ \frac{1}{(s^2 + 4)(s^2 + 1)} \right\}
$$

Use partial fractions decomposition:

$$
\frac{1}{(s^2 + 4)(s^2 + 1)} = \frac{A}{s^2 + 4} + \frac{B}{s^2 + 1}
$$

Solve for A and B:

$$
1 = A(s^2 + 1) + B(s^2 + 4)
$$

At s^2 terms:

$$
A + B = 0 \Rightarrow B = -A
$$

Constant terms:

$$
A \times 1 + B \times 4 = 1
$$

Substitute $B = -A$:

$$
A + (-A) \times 4 = 1 \Rightarrow A - 4A = 1 \Rightarrow -3A = 1 \Rightarrow A = -\frac{1}{3}
$$

Then:

$$
B = \frac{1}{3}
$$

Now, rewrite:

$$
\frac{1}{(s^2 + 4)(s^2 + 1)} = -\frac{1}{3} \frac{1}{s^2 + 4} + \frac{1}{3} \frac{1}{s^2 + 1}
$$

Inverse transforms:

$$
\mathcal{L}^{-1}\left\{ \frac{1}{s^2 + a^2} \right\} = \frac{\sin a t}{a}
$$

Thus:

$$
y_p(t) = -\frac{1}{3} \times \frac{\sin 2 t}{2} + \frac{1}{3} \times \sin t
$$

Simplify:

$$
y_p(t) = -\frac{1}{6} \sin 2 t + \frac{1}{3} \sin t
$$

Final Solution

Combining homogeneous and particular solutions:

$$
\boxed{
y(t) = y_0 \cos 2 t + \frac{y_1}{2} \sin 2 t - \frac{1}{6} \sin 2 t + \frac{1}{3} \sin t
}
$$

which simplifies to:

$$
y(t) = y_0 \cos 2 t + \left(\frac{y_1}{2} - \frac{1}{6} \right) \sin 2 t + \frac{1}{3} \sin t
$$

Broader Implications and Applications

Significance in Engineering and Physics

This methodology exemplifies the utility of Laplace transforms in diverse fields:

- Electrical Engineering: Circuit analysis with differential equations involving currents and voltages.
- Mechanical Engineering: Vibration analysis and control systems.
- Physics: Quantum mechanics, wave propagation, and heat transfer problems.

Limitations and Considerations

While the Laplace transform simplifies many problems, it requires:

- Functions to be of exponential order.
- Knowledge of inverse transforms, which sometimes involve complex partial fraction decompositions.
- Careful handling of initial conditions.

Numerical and Analytical Hybrids

In practice, solutions obtained via Laplace transforms often serve as a basis for numerical methods, especially when inverse transforms are complicated or involve special functions.

Concluding Remarks

The process of solving the initial-value problem \(Y'' + 4 y = \sin t \) via the Laplace transform underscores its elegance and efficiency. Transforming a differential equation into an algebraic equation streamlines the solution process, especially when initial conditions are specified. The method not only yields explicit solutions but also provides insights into the behavior of the system—such as oscillatory nature and transient responses.

This approach exemplifies the power of integral transforms in applied mathematics, bridging the gap between theoretical constructs and real-world problem-solving. As a robust technique, the Laplace transform continues to be a cornerstone in both academic and industrial applications, underpinning the analysis and design of complex dynamic systems.

References

- Oppenheim, A. V., Willsky, A. S., & Nawab, S.

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