Use The "mixed Partials" Check To See If The Following Differential Equation Is Exact. If It Is Exact

Use The "mixed Partials" Check To See If The Following Differential Equation Is Exact. If It Is Exact is a fundamental concept in the study of differential equations, particularly in the context of first-order equations. Determining whether a differential equation is exact is crucial because exact equations have straightforward methods for their solutions. This process involves examining the partial derivatives of the functions involved, specifically the mixed partial derivatives, to verify if certain conditions are satisfied. In this comprehensive guide, we will explore the concept of exact differential equations, the importance of the mixed partials check, how to perform this check, and what it means for the equation's solvability.

Understanding Differential Equations and Exactness

What Is a Differential Equation?

A differential equation is an equation that involves an unknown function and its derivatives. It expresses a relationship between the function and its derivatives and plays a key role in modeling real-world phenomena such as physics, engineering, biology, and economics. Differential equations can be classified into various types, including ordinary and partial differential equations, linear and nonlinear equations, and, importantly for our discussion, exact and non-exact equations.

What Does It Mean for an Equation to Be Exact?

An ordinary differential equation (ODE) of the form:

$$ M(x, y) \, dx + N(x, y) \, dy = 0 $$

is said to be exact if there exists a function $\Psi(x, y)$ such that:

$$ \frac{\partial \Psi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = N(x, y) $$

This means the differential form $M\,dx + N\,dy$ is an exact differential, derived from a potential function Ψ. When an equation is exact, it can be integrated directly to find solutions.

The Role of Mixed Partials in Determining Exactness

Clairaut’s Theorem and Mixed Partials

A key mathematical principle underpinning the check for exactness involves the equality of mixed partial derivatives. According to Clairaut’s theorem (or Schwarz's theorem), if the functions involved are continuous and sufficiently smooth, the mixed partial derivatives are equal:

$$\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}$$

This symmetry property is the foundation for the test of exactness in differential equations.

The Exactness Condition

Given the functions $M(x, y)$ and $N(x, y)$, the differential equation:

$$M(x, y) \, dx + N(x, y) \, dy = 0$$

is exact if and only if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This condition ensures that the potential function Ψ exists, satisfying the partial derivatives conditions.

Performing the Mixed Partials Check

Step-by-Step Procedure

To determine if a differential equation is exact using the mixed partials method, follow these steps:

1. Identify the functions $M(x, y)$ and $N(x, y)$ from the differential equation.
2. Calculate the partial derivative of M with respect to y: $\frac{\partial M}{\partial y}$.
3. Calculate the partial derivative of N with respect to x: $\frac{\partial N}{\partial x}$.
4. Compare the two derivatives. If they are equal at all points in the domain, the differential equation is exact.

Example Walkthrough

Suppose you have the differential equation:

$$ (2xy + y^2) \, dx + (x^2 + 2xy) \, dy = 0 $$

- Identify $M(x, y) = 2xy + y^2$
- Identify $N(x, y) = x^2 + 2xy$

Calculate the derivatives:

- $\frac{\partial M}{\partial y} = 2x + 2y$
- $\frac{\partial N}{\partial x} = 2x + 2y$

Since:

$$ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x + 2y $$

the equation is exact.

Implications of the Exactness Check

When the Equation Is Exact

If the mixed partials condition holds, the differential equation is exact, and you can find a potential function $\Psi(x, y)$. The solution process involves:

- Integrating $M(x, y)$ with respect to x to find $\Psi(x, y)$, including an arbitrary function of y.
- Differentiating this partial result with respect to y, equating it to $N(x, y)$, and solving for the unknown function of y.
- Combining these integrations to obtain the implicit solution $\Psi(x, y) = C$.

When the Equation Is Not Exact

If the derivatives are not equal, the differential equation is not exact. In such cases, methods to make the equation exact include:

- Finding an integrating factor, which can depend on x, y, or both.
- Multiplying through by the integrating factor to convert the equation into an exact form.

Using Integrating Factors to Achieve Exactness

What Are Integrating Factors?

An integrating factor is a function $\mu(x, y)$ that, when multiplied by the original differential equation, renders it exact. The concept is similar to integrating factors used in linear differential equations but applied here to ensure the exactness condition.

Common Strategies for Finding Integrating Factors

- If $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ depends only on x, an integrating factor $\mu(x)$ can be used.
- If the difference depends only on y, then $\mu(y)$ can be chosen.
- For more complex cases, more advanced methods or functions may be necessary.

Conclusion

Determining whether a differential equation is exact through the check of mixed partial derivatives is a powerful and fundamental technique in solving differential equations. By verifying if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

we can quickly identify whether the equation is directly solvable via potential functions. If the condition holds, the solution process becomes straightforward, involving integrations and partial derivatives. If not, the concept of integrating factors provides a pathway to transform the equation into an exact form, enabling solution strategies that would otherwise be inaccessible.

Understanding and applying the mixed partials check not only simplifies the problem-solving process but also deepens the comprehension of the underlying structure of differential equations. Whether you are studying for an exam, working on a real-world modeling problem, or exploring mathematical theory, mastering this check is essential for effective differential equation analysis.

Key Takeaways:
- The mixed partials check involves verifying the equality $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
- If the equation is exact, it can be integrated directly to find solutions.
- When not exact, integrating factors can be used to make the equation exact.
- Understanding the role of mixed partial derivatives is essential in differential equations and mathematical analysis.

By mastering these concepts, you will be better equipped to analyze and solve a wide range of differential equations encountered in mathematics, science, and engineering.

Frequently Asked Questions

What is the purpose of using the 'mixed partials' check in differential equations?

The purpose is to determine whether a differential equation is exact by verifying if the mixed partial derivatives of its components are equal.

How do you perform the 'mixed partials' check to see if a differential equation is exact?

You identify the functions $M(x,y)$ and $N(x,y)$ from the differential equation, then compute $\partial M/\partial y$ and $\partial N/\partial x$. If these are equal, the equation is exact.

What does it mean if $\partial M/\partial y$ equals $\partial N/\partial x$ in a differential equation?

It means the differential equation is exact, and there exists a potential function whose total differential matches the given equation.

Can a differential equation be non-exact initially but become exact after an integrating factor?

Yes, sometimes multiplying by an integrating factor transforms a non-exact differential equation into an exact one, facilitating its solution.

What are common signs that a differential equation might be exact?

If the differential equation can be written in the form $M(x,y)dx + N(x,y)dy = 0$ and the functions are sufficiently smooth, checking the mixed partials is a standard method to verify exactness.

Why is verifying the equality of mixed partial derivatives essential in solving differential equations?

Because this equality confirms the existence of a potential function, allowing the solution to be found via integration, simplifying the process.

What should you do if the mixed partials are not equal in a differential equation?

The equation is not exact, and you may need to find an integrating factor or use other methods like substitution to solve it.

Is the mixed partials check sufficient to determine if a differential equation is exact?

Yes, checking if $\partial M/\partial y$ equals $\partial N/\partial x$ is the standard and sufficient test for exactness in first-order differential equations.

What are the steps after confirming a differential equation is

exact using the mixed partials check?

Once confirmed, integrate M with respect to x (or N with respect to y), then find the potential function and solve for the solution accordingly.

Additional Resources

Using the "Mixed Partials" Check to Determine if a Differential Equation Is Exact

Understanding whether a differential equation is exact is a fundamental step in solving many first-order differential equations. This process hinges on analyzing the structure of the equation, particularly the relationship between its partial derivatives. The method of testing whether an equation is exact involves examining the mixed partial derivatives of its constituent functions—a technique that provides insight into whether a potential solution function exists. This article delves into the concept of exact differential equations, explains the significance of mixed partial derivatives, and provides a comprehensive guide to applying the "mixed partials" check to determine exactness.

Introduction to Exact Differential Equations

What Are Differential Equations?

Differential equations are equations involving derivatives of an unknown function. They’re pervasive across scientific disciplines, modeling phenomena from physics and engineering to biology and economics. First-order differential equations, which involve derivatives of the first order, are among the most fundamental and widely studied types.

Defining Exact Differential Equations

A first-order differential equation can often be written in the form:

$$ M(x, y) \, dx + N(x, y) \, dy = 0 $$

where $M(x, y)$ and $N(x, y)$ are functions of x and y.

An equation of this form is called exact if there exists a potential function $\Psi(x, y)$ such that:

$$ d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy = 0 $$

Matching terms, we see that:

$$ \frac{\partial \Psi}{\partial x} = M(x, y) $$

$$\frac{\partial \Psi}{\partial y} = N(x, y)$$

The core criterion for exactness is that the mixed partial derivatives of Ψ with respect to x and y are equal, owing to Clairaut’s theorem (which states that, under suitable smoothness conditions, mixed partial derivatives are equal):

$$\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial^2 \Psi}{\partial y \partial x}$$

which translates to:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This condition forms the basis of the "mixed partials" check.

The "Mixed Partials" Check: Concept and Significance

Understanding the Partial Derivatives

In the context of the differential equation $M(x, y) dx + N(x, y) dy = 0$, the functions M and N are known as the coefficient functions. To determine if the equation is exact, we analyze their partial derivatives:

- $\frac{\partial M}{\partial y}$: the rate of change of M with respect to y, holding x constant.
- $\frac{\partial N}{\partial x}$: the rate of change of N with respect to x, holding y constant.

Why Check the Equality of Mixed Partials?

The equality $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is rooted in calculus theory, specifically Clairaut’s theorem. When these derivatives are equal, it indicates that the functions M and N are closely linked, suggesting the existence of a potential function Ψ. If such a Ψ exists, the differential equation can be integrated to find an explicit solution.

This check is both necessary and sufficient for the equation to be exact—meaning:

- If the equality holds, the differential equation is exact.
- If not, the equation is not exact, although it might be made exact through an integrating factor.

Practical Importance of the Check

Performing the "mixed partials" check is a straightforward step in the solution process:

- It quickly determines whether the existing form of the differential equation can be integrated directly.
- It helps avoid unnecessary computational effort, guiding the solver toward the appropriate method—either direct integration or seeking an integrating factor.

Step-by-Step Procedure for the "Mixed Partials" Check

1. Write the Differential Equation in Standard Form

Ensure your differential equation is in the form:

$$ M(x, y) \, dx + N(x, y) \, dy = 0 $$

or equivalently,

$$ \frac{dy}{dx} = - \frac{M(x, y)}{N(x, y)} $$

2. Identify the Functions M and N

Extract the functions M and N from the equation. For example, if your equation is:

$$ (2xy + y^2) dx + (x^2 + 2xy) dy = 0 $$

then:

$$ M(x, y) = 2xy + y^2 $$
$$ N(x, y) = x^2 + 2xy $$

3. Compute the Partial Derivatives

Calculate:

$$ \frac{\partial M}{\partial y} $$
$$ \frac{\partial N}{\partial x} $$

Using the example:

- $\frac{\partial M}{\partial y} = 2x + 2y$
- $\frac{\partial N}{\partial x} = 2x + 2y$

4. Compare the Partial Derivatives

Check whether:

$$ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} $$

If the derivatives are equal, the differential equation is exact.

5. Confirm and Proceed

- If equal: The equation is exact; proceed to find the potential function Ψ.
- If not: The equation is not exact; consider whether an integrating factor can be applied or if alternative solution methods are necessary.

Illustrative Examples

Example 1: An Exact Differential Equation

Given:

$$ (3xy + 2) dx + (x^2 + y) dy = 0 $$

Identify:

- $M(x, y) = 3xy + 2$
- $N(x, y) = x^2 + y$

Calculate:

$$ \frac{\partial M}{\partial y} = 3x $$
$$ \frac{\partial N}{\partial x} = 2x $$

Since:

$$ 3x \neq 2x $$

the partial derivatives are not equal, so the equation is not exact. Further steps might involve finding an integrating factor.

Example 2: An Exact Differential Equation

Given:

$$ (2xy + y^2) dx + (x^2 + 2xy) dy = 0 $$

Identify:

- $M(x, y) = 2xy + y^2$
- $N(x, y) = x^2 + 2xy$

Calculate:

$$ \frac{\partial M}{\partial y} = 2x + 2y $$
$$ \frac{\partial N}{\partial x} = 2x + 2y $$

Since:

$$ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} $$

the equation is exact.

Implications and Further Steps After Confirming Exactness

Finding the Potential Function Ψ

Once an equation is verified to be exact, the next step involves:

1. Integrating $M(x, y)$ with respect to x to find $\Psi(x, y)$:

$$ \Psi(x, y) = \int M(x, y) dx + h(y) $$

where $h(y)$ is an arbitrary function of y.

2. Differentiating $\Psi(x, y)$ with respect to y:

$$ \frac{\partial \Psi}{\partial y} = N(x, y) $$

3. Solving for $h(y)$ by equating the derivatives.

4. The solution to the differential equation is then:

$$\Psi(x, y) = C$$

where C is an arbitrary constant.

Handling Non-Exact Equations

If the check reveals that the equation is not exact, various techniques can be employed:

- Finding an integrating factor: A function $\mu(x, y)$ that, when multiplied through, makes the equation exact.
- Alternative solution methods: Such as substitution, Bernoulli’s equation, or numerical approaches.

Conclusion: The Power and Precision of the "Mixed Partials" Check

The "mixed partials" check stands as a cornerstone in the analysis of first-order differential equations. Its elegance lies in leveraging fundamental calculus principles to inform the solv

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