

Use Upper And Lower Sums To Approximate The Area Of The Region Using The Given Number Of Subintervals

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Calculating the exact area under a curve between two points can often be a challenging task, especially when dealing with complex functions. To address this, mathematicians have developed methods such as Riemann sums, which provide approximate values for the area of a region bounded by a curve, the x-axis, and two vertical lines. Among these, the concepts of upper and lower sums are fundamental tools that help in approximating the area with a desired level of accuracy by dividing the interval into smaller subintervals. This article explores how to use upper and lower sums to approximate the area of a region, emphasizing the importance of the number of subintervals used and how it influences the approximation's accuracy.

Understanding the Basics: Riemann Sums and Their Role

What Are Riemann Sums?

Riemann sums are a way to approximate the area under a curve by dividing the region into smaller sections, calculating the sum of areas of simple geometric shapes (usually rectangles), and then refining these sums to approach the exact area. The general idea involves partitioning the interval of interest into subintervals, choosing sample points within each subinterval, and summing up the areas of rectangles constructed using these points.

Why Use Upper and Lower Sums?

Upper and lower sums are specific types of Riemann sums that provide bounds on the area under the curve:

- Lower Sum (L): Uses the minimum function value within each subinterval, resulting in a sum that underestimates the true area.
- Upper Sum (U): Uses the maximum function value within each subinterval, resulting in a sum that overestimates the true area.

As the number of subintervals increases, these bounds become tighter, and the sums converge to the actual area.

Dividing the Interval: Subintervals and Their Significance

Partitioning the Interval

Suppose you want to find the area under a curve $f(x)$ from a to b . The first step is to divide the interval $[a, b]$ into smaller subintervals:

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

Each subinterval is $[x_{i-1}, x_i]$, where $i = 1, 2, \dots, n$.

The choice of the number of subintervals, n , affects the accuracy of the approximation:

- Increasing n (making subintervals narrower) generally improves the approximation.
- Decreasing n leads to less accurate estimates but simplifies calculations.

Width of Subintervals

The width of each subinterval, denoted as Δx_i , is computed as:

$$\Delta x_i = x_i - x_{i-1}$$

If the subintervals are evenly spaced, then:

$$\Delta x = \frac{b - a}{n}$$

Uniform partitions are easier to work with, but non-uniform partitions can be used for functions with varying behavior.

Constructing Upper and Lower Sums

Calculating the Lower Sum

The lower sum, L_n , is obtained by summing the areas of rectangles where each rectangle's

height is the minimum value of the function within the subinterval:

$$L_n = \sum_{i=1}^n m_i \Delta x_i$$

where:

$$m_i = \min_{x \in [x_{i-1}, x_i]} f(x)$$

This sum underestimates the true area because it uses the smallest function value in each subinterval.

Calculating the Upper Sum

Similarly, the upper sum, U_n , is calculated using the maximum value of the function within each subinterval:

$$U_n = \sum_{i=1}^n M_i \Delta x_i$$

where:

$$M_i = \max_{x \in [x_{i-1}, x_i]} f(x)$$

This sum overestimates the true area because it uses the largest function value in each subinterval.

Using the Sums to Approximate Area

Step-by-Step Approach

1. Select the number of subintervals (n): Decide how many subdivisions you will use. Typically, starting with a small n and increasing it improves accuracy.
2. Partition the interval: Divide $[a, b]$ into n subintervals, either uniformly or non-uniformly.
3. Determine the minimum and maximum values: For each subinterval, find the minimum and maximum values of $f(x)$.
4. Compute lower and upper sums: Use the formulas for L_n and U_n to calculate the sums.
5. Refine the approximation: Increase n and repeat the process. The more subintervals used, the closer the sums are to the actual area.

6. Estimate the area: When the difference between the upper and lower sums becomes negligible, you can take their average as a good approximation of the area.

Mathematical Significance

As $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} U_n = \text{the exact area under the curve}$$

This convergence is the foundation of definite integrals in calculus.

Practical Examples and Applications

Example: Approximate Area Under $f(x) = x^2$ from 0 to 1

Suppose you want to approximate the area under $f(x) = x^2$ from $a = 0$ to $b = 1$ using $n = 4$ subintervals.

1. Partition the interval:

$$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1$$

2. Calculate $\Delta x = 0.25$.

3. Find minima and maxima in each subinterval:

Subinterval	$f(x)$ at endpoints	Min $f(x)$	Max $f(x)$
[0, 0.25]	$f(0)=0, f(0.25)=0.0625$	0	0.0625
[0.25, 0.5]	$f(0.25)=0.0625, f(0.5)=0.25$	0.0625	0.25
[0.5, 0.75]	$f(0.5)=0.25, f(0.75)=0.5625$	0.25	0.5625
[0.75, 1]	$f(0.75)=0.5625, f(1)=1$	0.5625	1

4. Calculate the lower sum:

$$L_4 = \sum_{i=1}^4 m_i \Delta x = (0 \times 0.25) + (0.0625 \times 0.25) + (0.25 \times 0.25) + (0.5625 \times 0.25) = 0 + 0.015625 + 0.0625 + 0.140625 = 0.21875$$

5. Calculate the upper sum:

$$U_4 = \sum_{i=1}^4 M_i \Delta x = (0.0625 \times 0.25) + (0.25 \times 0.25) + (0.5625 \times 0.25) + (1 \times 0.25) = 0.015625 + 0.0625 + 0.140625 + 0.25 = 0.46875$$

6. Observation: The true area lies between 0.21875 and 0.46875. Increasing (n) will narrow this interval.

Advantages and Limitations of Using Upper and Lower Sums

Advantages

- Provides bounds: Upper and lower sums give a range within which the true area lies.
- Improves with more subintervals: Increasing (n) makes the sums converge to the exact integral.
- Foundational for calculus: These concepts underpin the definition of the Riemann integral.

Limitations

- Computational effort: Increasing the number of subintervals can be computationally intensive.
- Function complexity: For highly oscillatory or complex functions, finding minima and maxima within subintervals can be challenging

Frequently Asked Questions

What are upper and lower sums in the context of approximating the area under a curve?

Upper and lower sums are methods of approximating the area under a curve by dividing the region into subintervals. The lower sum uses the minimum function value in each subinterval, while the upper sum uses the maximum, providing bounds for the true area.

How do you compute the lower sum for a given number of subintervals?

To compute the lower sum, divide the interval into equal subintervals, find the minimum function value in each subinterval, then multiply each by its subinterval width and sum all these products.

How do you compute the upper sum for a given number of subintervals?

To compute the upper sum, divide the interval into equal subintervals, find the maximum function value in each subinterval, then multiply each by its subinterval width and sum all these products.

Why is increasing the number of subintervals beneficial when using upper and lower sums?

Increasing the number of subintervals makes the upper and lower sums more accurate as the subintervals become smaller, causing the sums to better approximate the true area under the curve.

Can upper and lower sums be used to determine the exact area under a curve?

No, upper and lower sums provide bounds for the area. As the number of subintervals increases indefinitely, the sums converge to the exact area, which is the definite integral of the function.

What is the significance of choosing the same number of subintervals for both upper and lower sums?

Using the same number of subintervals allows for a direct comparison between the upper and lower bounds, which helps in estimating the accuracy of the approximation and understanding how close the sums are to the true area.

How does the choice of subintervals affect the accuracy of the approximation using upper and lower sums?

Uniform subintervals simplify calculations and improve accuracy as their size decreases. Non-uniform subintervals can be used to focus on regions with higher curvature, but generally, smaller and more evenly spaced subintervals lead to better approximations.

What is the relationship between the number of subintervals and the limit of the upper and lower sums?

As the number of subintervals approaches infinity, both the upper and lower sums converge to the same value, which is the exact area under the curve, representing the definite integral of the function.

Additional Resources

Use Upper And Lower Sums To Approximate The Area Of The Region Using The Given Number Of Subintervals

Understanding how to approximate the area under a curve is a fundamental concept in integral

calculus, bridging the gap between discrete and continuous analysis. The method of upper and lower sums is a classical approach that provides a systematic way to estimate the area of a region under a curve, especially when the exact integral is challenging to compute directly. This technique involves subdividing the interval into smaller subintervals, calculating sums based on the supremum (least upper bound) and infimum (greatest lower bound) of the function on each subinterval, and then refining these estimates by increasing the number of subintervals.

This comprehensive review explores the theoretical foundation and practical application of upper and lower sums, emphasizing their role in approximating areas with a specified number of subintervals. We will delve into the concepts, calculations, properties, and significance of these sums in the broader context of integral approximation, providing detailed insights suitable for students, educators, and enthusiasts.

Foundations of Area Approximation: The Rationale Behind Upper and Lower Sums

Before diving into the mechanics of upper and lower sums, it's essential to understand why these methods are valuable in the context of area approximation.

The Need for Approximation

- Exact Integration Challenges: Many functions do not have elementary antiderivatives or are difficult to integrate analytically.
- Practical Applications: Engineers, scientists, and mathematicians often need approximate solutions to real-world problems where the precise area under a curve corresponds to physical quantities like distance, volume, or probability.
- Numerical Integration: Approximate methods like Riemann sums, including upper and lower sums, serve as foundational techniques for numerical integration.

The Concept of Partitioning

- The core idea is to divide the interval $[a, b]$, over which the area is to be computed, into smaller subintervals.
- Increasing the number of subintervals generally leads to a better approximation, approaching the exact area as the number of subintervals tends to infinity.

Introduction to Upper and Lower Sums

- These sums are based on selecting specific values of the function within each subinterval:
- The supremum (least upper bound) of the function on each subinterval for the upper sum.
- The infimum (greatest lower bound) of the function on each subinterval for the lower sum.
- By summing these values times the width of the subintervals, we create bounds that contain the true area.

Understanding the Construction of Upper and Lower Sums

The process involves several key steps, which we will explore in detail.

Step 1: Selecting a Partition

- A partition P of the interval $[a, b]$ is a finite set of points:

$$P = \{x_0, x_1, x_2, \dots, x_n\}$$

where

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

- The subintervals are $[x_{k-1}, x_k]$ for $(k=1, 2, \dots, n)$.

- The number of subintervals (n) is given, and the length of each subinterval is:

$$\Delta x_k = x_k - x_{k-1}$$

- For simplicity, many analyses assume equal subintervals, but the method extends to uneven partitions.

Step 2: Determining Supremum and Infimum of the Function

- For each subinterval $[x_{k-1}, x_k]$, identify:

- The supremum (M_k) , which is the least upper bound of the function (f) on that subinterval.

- The infimum (m_k) , which is the greatest lower bound of (f) on that subinterval.

- These are mathematically expressed as:

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x)$$

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x)$$

\]

- In practice, if f is continuous and well-behaved, these bounds are simply the minimum and maximum values of f on each subinterval.

Step 3: Calculating the Upper and Lower Sums

- Lower Sum (L_n):

\[

$$L_n = \sum_{k=1}^n m_k \Delta x_k$$

\]

This sum uses the minimum value of f on each subinterval, resulting in an underestimate of the total area if f is positive.

- Upper Sum (U_n):

\[

$$U_n = \sum_{k=1}^n M_k \Delta x_k$$

\]

This sum uses the maximum value of f on each subinterval, providing an overestimate of the area.

- Both sums are Riemann sums but distinguished by their choice of bounds for the function within each subinterval.

Step 4: Interpreting the Sums as Bounds

- The true area under the curve, denoted as A , satisfies:

\[

$$L_n \leq A \leq U_n$$

\]

- As the number of subintervals n increases, the bounds become tighter:

\[

$$\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} U_n = A$$

\]

- This convergence hinges on the continuity of f , but even for discontinuous functions, the sums can provide useful bounds.

Mathematical Properties and Theoretical Significance

The utility of upper and lower sums is deeply rooted in their mathematical properties, which

underpin the theory of the Riemann integral.

Monotonicity of Sums with Increasing Subintervals

- For a fixed partition, the lower sums form a non-decreasing sequence as the partition is refined (more subintervals):

$$\forall L_n \leq L_{n+1}$$

- Conversely, the upper sums form a non-increasing sequence:

$$\forall U_n \geq U_{n+1}$$

- These properties ensure that as we increase the number of subintervals, the bounds approach each other.

Refinement of Partitions and Convergence

- A refinement involves subdividing existing subintervals into smaller parts, leading to new partitions (P') :

$$P \subset P'$$

- The refinement satisfies:

$$L(P) \leq L(P') \leq U(P') \leq U(P)$$

- As the maximum subinterval length $(\max \Delta x_k \rightarrow 0)$, the difference between upper and lower sums diminishes, and the sums converge to the true area.

Riemann Integrability and Sum Equivalence

- A function (f) on $([a, b])$ is Riemann integrable if and only if:

$$\lim_{n \rightarrow \infty} U_n = \lim_{n \rightarrow \infty} L_n = A$$

- The common limit (A) is called the Riemann integral of (f) .

- The upper and lower sums provide a practical method for approximating the integral and verifying integrability.

Practical Computation and Examples

Applying the theory involves concrete calculations. Here, we explore typical steps with examples illustrating how to perform upper and lower sum approximations.

Example 1: Approximating the Area under $(f(x) = x^2)$ on $([0,1])$

- Given: $(f(x) = x^2)$, interval $([0,1])$, number of subintervals (n) .

- Partition: Equal subintervals with width:

$$\Delta x = \frac{b - a}{n} = \frac{1}{n}$$

$$x_k = \frac{k}{n} \quad \text{for } k=0,1,\dots,n$$

- Supremum and infimum on each subinterval:

$$m_k = \left(\frac{k-1}{n} \right)^2$$

$$M_k = \left(\frac{k}{n} \right)^2$$

- Lower sum:

$$L_n = \sum_{k=1}^n m_k \Delta x = \sum_{k=1}^n \left(\frac{k-1}{n} \right)^2 \frac{1}{n}$$

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