Verify That The Divergence Theorem Is True For The Vector Field F(x,y,z)=3xi+xyj+3xzk On The Region E

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Introduction to the Divergence Theorem

The Divergence Theorem, also known as Gauss's Divergence Theorem, plays a fundamental role in vector calculus. It provides a powerful way to relate the flux of a vector field across a closed surface to the divergence of the field within the volume enclosed by that surface. This theorem simplifies many calculations in physics and engineering, especially in fluid dynamics, electromagnetism, and other fields involving vector fields.

In this article, we aim to verify the Divergence Theorem for a specific vector field, F(x, y, z) = 3x i + xy j + 3xz k, over a certain region E. Our goal is to demonstrate, step-by-step, that the theorem holds by computing both the surface integral of the flux and the volume integral of divergence, confirming their equality.

Understanding the Vector Field F(x, y, z)

Before delving into the verification process, it’s essential to understand the components of the vector field:

- F(x, y, z) = 3x i + xy j + 3xz k

This vector field has three components:
- $F_x = 3x$
- $F_y = xy$
- $F_z = 3xz$

These components suggest interactions between the variables, representing a flow or field that varies with position in space.

Defining the Region E

The verification process requires specifying the region E over which the divergence theorem is applied. Common choices include:

- A rectangular box (parallelepiped)
- A sphere
- A cylinder

For the purposes of this demonstration, assume E is the unit cube defined by:

$$
E = \{ (x, y, z) | 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1 \}
$$

This choice simplifies calculations while still illustrating the core principles of the divergence theorem.

Step 1: Computing the Divergence of F

The divergence of a vector field F = $F_x i + F_y j + F_z k$ is given by:

$$
\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}
$$

Calculating each term:

- $\frac{\partial F_x}{\partial x} = \frac{\partial (3x)}{\partial x} = 3$
- $\frac{\partial F_y}{\partial y} = \frac{\partial (xy)}{\partial y} = x$
- $\frac{\partial F_z}{\partial z} = \frac{\partial (3xz)}{\partial z} = 3x$

Adding these:

$$
\nabla \cdot F = 3 + x + 3x = 3 + 4x
$$

Thus, the divergence of F is:

$$
\nabla \cdot F = 3 + 4x
$$

Step 2: Volume Integral of Divergence over E

According to the Divergence Theorem:

$$
\iint_{S} F \cdot n \, dS = \iiint_{E} (\nabla \cdot F) \, dV
$$

where S is the boundary surface of E.

Calculate the volume integral:

$$
\iiint_{E} (3 + 4x) \, dV
$$

Since E is the unit cube, integrate over x, y, z:

$$
\int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (3 + 4x) \, dx \, dy \, dz
$$

Step-by-step:

1. Integrate with respect to x:

$$
\int_{0}^{1} (3 + 4x) \, dx = \left[3x + 2x^2 \right]_0^1 = (3 \times 1 + 2 \times 1^2) - 0 = 3 + 2 = 5
$$

2. Integrate with respect to y:

$$
\int_{0}^{1} 1 \, dy = 1
$$

3. Integrate with respect to z:

$$
\int_{0}^{1} 1 \, dz = 1
$$

Putting it all together:

$$
\iiint_{E} (\nabla \cdot F) \, dV = 5 \times 1 \times 1 = 5
$$

The volume integral of divergence over E equals 5.

Step 3: Computing the Surface Integral of F · n over S

The boundary surface S consists of six faces:

- $x=0$ (left face)
- $x=1$ (right face)
- $y=0$ (front face)
- $y=1$ (back face)
- $z=0$ (bottom face)
- $z=1$ (top face)

We compute the flux across each face separately, then sum the results.

Face 1: $x=0$

- Outward normal: $n = -i$
- $F = (3x, xy, 3xz)$
- At $x=0$: $F = (0, 0, 0)$

Flux:

$$
\iint_{x=0} F \cdot n \, dS = \iint_{y=0}^{1} \int_{z=0}^{1} (0,0,0) \cdot (-i) \, dy\, dz = 0
$$

Face 2: $x=1$

- Outward normal: $n = i$
- $F = (3, y, 3z)$

Flux:

$$
\iint_{x=1} F \cdot n \, dS = \iint_{y=0}^{1} \int_{z=0}^{1} (3, y, 3z) \cdot i \, dy\, dz = \iint (3) \, dy\, dz
$$

Calculate:

$$
\int_{0}^{1} \int_{0}^{1} 3 \, dy\, dz = 3 \times 1 \times 1 = 3
$$

Face 3: $y=0$

- Outward normal: $n = -j$
- $F = (3x, xy, 3xz)$
- At $y=0$: $F = (3x, 0, 3xz)$

Flux:

$$
F \cdot n = (3x, 0, 3xz) \cdot (0, -1, 0) = 0
$$

Integral over the face:

$$
0
$$

Face 4: $y=1$

- Outward normal: $n= j$
- $F = (3x, x, 3xz)$

Flux:

$$
F \cdot n = (3x, x, 3xz) \cdot (0, 1, 0) = x
$$

Integral:

$$
\int_{x=0}^{1} \int_{z=0}^{1} x \, dx\, dz
$$

Calculate:

$$
\int_{0}^{1} x \, dx = \frac{1}{2}
$$

and

$$
\int_{0}^{1} 1 \, dz = 1
$$

Total:

$$
\frac{1}{2} \times 1 = \frac{1}{2}
$$

Face 5: $z=0$

- Outward normal: $n = -k$
- $F = (3x, xy, 0)$

Flux:

\[
F \cdot n = (3x, xy, 0) \cdot (0, 0, -1) = 0
\]

Integral is zero.

Face 6: \(z=1\)

- Outward normal: \(n= k\)
- \(F= (3x, xy, 3x)\)

Flux:

\[
F \cdot n = (3x, xy, 3x) \cdot (0, 0, 1) = 3x
\]

Integral:

\[
\int_{x=0}^{1} \int_{y=0}^{1} 3x \, dy\, dx
\]

Calculate:

\[
\

Frequently Asked Questions

How do we verify the Divergence Theorem for the vector field F(x,y,z) = 3x i + xy j + 3xz k over a region E?

To verify the Divergence Theorem, compute the flux of F across the boundary surface of E and compare it to the triple integral of the divergence of F over E. If both are equal, the theorem holds for this field and region.

What is the divergence of the vector field F(x,y,z) = 3x i + xy j + 3xz k?

The divergence of F is $\nabla \cdot F = \partial/\partial x\ (3x) + \partial/\partial y\ (xy) + \partial/\partial z\ (3xz) = 3 + x + 3x = 3 + x + 3x = 3 + 4x$.

How do we set up the volume integral of the divergence to verify the Divergence Theorem for this field?

We integrate $\nabla\cdot F = 3 + 4x$ over the region E, which involves defining the bounds of E and computing the integral $\iiint_E (3 + 4x)\, dV$.

What steps are involved in computing the surface integral of F over the boundary of E?

Parametrize each part of the boundary surface, compute the surface element dS, evaluate $F \cdot n$ over each surface, and sum the flux contributions for the entire boundary.

Why is verifying the Divergence Theorem important for this vector field and region?

Verifying the theorem provides a consistency check for the calculations and confirms that the divergence theorem holds for the specific vector field and region E, ensuring the correctness of flux and divergence computations.

Additional Resources

Divergence Theorem Verification for the Vector Field $\mathbf{F}(x,y,z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 3xz\,\mathbf{k}$ on the Region E

Introduction to the Divergence Theorem: A Mathematical Powerhouse

The Divergence Theorem, also known as Gauss's Divergence Theorem, is a fundamental result in vector calculus that bridges the gap between the flux of a vector field across a closed surface and the behavior of its divergence within the volume enclosed by that surface. This theorem is invaluable for simplifying complex flux calculations, making it a staple tool in physics, engineering, and mathematics.

At its core, the theorem states:

$$
\iint_{S} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_{V} (\nabla \cdot \mathbf{F})\, dV
$$

where:
- S is a closed surface bounding the volume V,
- $\mathbf{n}$ is the outward-pointing unit normal vector on S,
- $\mathbf{F}$ is a continuously differentiable vector field,

- $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$.

Our task is to verify that the Divergence Theorem holds for the specific vector field:

$$
\mathbf{F}(x,y,z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 3xz\,\mathbf{k}
$$

over a particular region E. Although the original problem does not specify the shape of E, typical choices include common geometric regions like spheres, cubes, or cylinders. For this review, we'll consider a representative region—say, the unit cube $0 \leq x,y,z \leq 1$—to clearly demonstrate the process. This approach offers clarity and generalizable methodology.

Understanding the Region E

Before delving into calculations, it’s crucial to define the region E. For the purposes of this verification, assume:

Region E: The unit cube in the first octant, defined by:

$$
E = \{ (x,y,z) \mid 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1 \}
$$

This choice simplifies boundary calculations, as the surface S consists of six faces, each aligned with coordinate planes.

Step 1: Computing the Divergence $\nabla \cdot \mathbf{F}$

The divergence of a vector field $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ is given by:

$$
\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
$$

Given:

$$
P = 3x, \quad Q = xy, \quad R = 3xz
$$

\]

Compute each partial derivative:

1. $\displaystyle \frac{\partial P}{\partial x} = \frac{\partial}{\partial x} (3x) = 3$

2. $\displaystyle \frac{\partial Q}{\partial y} = \frac{\partial}{\partial y} (xy) = x$

3. $\displaystyle \frac{\partial R}{\partial z} = \frac{\partial}{\partial z} (3xz) = 3x$

Adding these:

$$
\nabla \cdot \mathbf{F} = 3 + x + 3x = 3 + 4x
$$

This divergence function is straightforward: it depends linearly on x, which simplifies volume integration.

Step 2: Computing the Volume Integral $\iiint_E (\nabla \cdot \mathbf{F})\, dV$

The divergence theorem states that:

$$
\iint_{S} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E (3 + 4x) \, dV
$$

Let's evaluate the volume integral over the unit cube $[0,1]^3$:

$$
\iiint_E (3 + 4x) \, dV = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (3 + 4x) \, dx\, dy\, dz
$$

Since the integrand does not depend on y or z, integration over these variables is straightforward:

$$
= \left(\int_{z=0}^{1} dz \right) \left(\int_{y=0}^{1} dy \right) \left(\int_{x=0}^{1} (3 + 4x) dx \right)
$$

Calculations:

- $\int_{z=0}^{1} dz = 1$
- $\int_{y=0}^{1} dy = 1$

Now, the \(x \)-integral:

\[
\int_0^1 (3 + 4x) dx = \int_0^1 3 dx + \int_0^1 4x dx = 3 \times 1 + 4 \times \frac{1}{2} = 3 + 2 = 5
\]

Thus, the volume integral:

\[
\boxed{
\text{Volume flux} = 1 \times 1 \times 5 = 5
}
\]

This result indicates that the total "outflow" of \(\mathbf{F} \) inside the cube, as predicted by the divergence theorem, is 5.

Step 3: Computing the Surface Flux \(\iint_{S} \mathbf{F} \cdot \mathbf{n}\, dS \)

To verify the theorem's validity, compute the flux across each face of the cube and sum these contributions.

The cube has six faces:

1. Face at \(x=0 \): Normal vector \(\mathbf{n} = -\mathbf{i} \)
2. Face at \(x=1 \): Normal vector \(\mathbf{n} = \mathbf{i} \)
3. Face at \(y=0 \): Normal vector \(\mathbf{n} = -\mathbf{j} \)
4. Face at \(y=1 \): Normal vector \(\mathbf{n} = \mathbf{j} \)
5. Face at \(z=0 \): Normal vector \(\mathbf{n} = -\mathbf{k} \)
6. Face at \(z=1 \): Normal vector \(\mathbf{n} = \mathbf{k} \)

Flux across \(x=1 \) face:

- \(y,z \in [0,1] \), \(x=1 \)
- \(\mathbf{F} = (3 \times 1, 1 \times y, 3 \times 1 \times z) = (3, y, 3z) \)
- Outward normal \(\mathbf{n} = \mathbf{i} \)

Flux:

\[

\iint_{x=1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{0}^{1} \iint_{0}^{1} (3, y, 3z) \cdot (1, 0, 0) \, dy\, dz = \iint_{0}^{1} \iint_{0}^{1} 3 \, dy\, dz
\]

\[
= 3 \times (1 \times 1) = 3
\]

Flux across \(x=0 \) face:

- \(x=0 \), \(y,z \in [0,1] \)
- \(\mathbf{F} = (0, 0, 0) \) (since \(P=0 \), \(Q=0 \), \(R=0 \))
- Outward normal \(\mathbf{n} = -\mathbf{i} \)

Flux:

\[
\iint_{x=0} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{0}^{1} \iint_{0}^{1} (0, y, 3z) \cdot (-1, 0, 0) \, dy\, dz = 0
\]

Flux across \(y=1 \) face:

- \(y=1 \), \(x,z \in [0,1] \)
- \(\mathbf{F} = (3x, x \times 1, 3x z) = (3x, x, 3x z) \)
- Normal \(\

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