Verify That The Function Satishes The Three Hypotheses Of Rolle's Theorem On The Given Interval. Then

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In calculus, Rolle's Theorem is a fundamental result that guarantees the existence of at least one point where the derivative of a function is zero, given certain conditions are met. Before applying Rolle's Theorem to a particular function over a specific interval, it is essential to verify that all three of its hypotheses are satisfied. Doing so ensures the theorem's conclusion is valid and applicable. This comprehensive guide will walk you through the process of verifying these hypotheses systematically, providing clarity and confidence in your calculus applications.

Understanding Rolle's Theorem

Rolle's Theorem states that if a function $f(x)$ meets certain conditions on a closed interval $[a, b]$, then there exists at least one point $c \in (a, b)$ where the derivative $f'(c)$ is zero. Formally:

> If all of the following conditions are met:
> 1. $f(x)$ is continuous on $[a, b]$,
> 2. $f(x)$ is differentiable on (a, b),
> 3. $f(a) = f(b)$,
>
> then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Understanding these hypotheses is crucial because they underpin the theorem's conclusion. Failure to verify any of these can lead to incorrect assumptions about the behavior of the function.

Step-by-Step Process to Verify the Hypotheses

To verify that a function satisfies the three hypotheses of Rolle's Theorem on a given interval, follow these steps:

1. Check Continuity on the Closed Interval [a, b]

Why it matters: Continuity ensures there are no gaps, jumps, or asymptotes disrupting the function's behavior, which is essential for the theorem's validity.

How to verify:

- Identify the function's domain: Ensure that the function is defined at every point in $[a, b]$.
- Examine the function's form: Functions composed of polynomials, rational functions (where denominators are not zero), root functions (with non-negative radicands), exponential, and trigonometric functions are generally continuous on their domains.
- Check for discontinuities: Look for points where the function might be undefined or have jumps, such as division by zero, square roots of negative numbers, or other discontinuities.

Example:

Suppose $f(x) = x^3 - 3x + 2$ on $[1, 3]$.

- The function is a polynomial, which is continuous everywhere.
- Therefore, $f(x)$ is continuous on $[1, 3]$.

2. Check Differentiability on the Open Interval (a, b)

Why it matters: Differentiability implies the function is smooth (no sharp corners or cusps) within the interval, which is necessary for the existence of a derivative at some point.

How to verify:

- Identify the function's form: Polynomial functions are differentiable everywhere; rational functions are differentiable where the denominator is not zero; root and exponential functions are differentiable everywhere in their domains.
- Check for points of non-differentiability: Look for corners, cusps, vertical tangents, or discontinuities within (a, b).

Example:

Using the previous function $f(x) = x^3 - 3x + 2$ on $[1, 3]$:

- Since it's a polynomial, it's differentiable everywhere.
- Thus, it is differentiable on $(1, 3)$.

3. Verify that $f(a) = f(b)$

Why it matters: The function must have the same value at the endpoints for Rolle's Theorem to guarantee a point where the derivative is zero.

How to verify:

- Calculate $f(a)$ and $f(b)$: Substitute the endpoints into the function.
- Compare the results: Confirm whether $f(a) = f(b)$.

Example:

Check $f(1)$ and $f(3)$:

- $f(1) = 1^3 - 3(1) + 2 = 1 - 3 + 2 = 0$.
- $f(3) = 3^3 - 3(3) + 2 = 27 - 9 + 2 = 20$.

Since $f(1) \neq f(3)$, Rolle's Theorem does not apply to this function on $[1, 3]$.

Important: If the values at the endpoints are equal, then the third hypothesis is satisfied; otherwise, the theorem cannot be applied directly.

Applying the Verification to an Example Function

Let's consider a concrete example to illustrate the process fully.

Given:

- $f(x) = \cos x$,
- Interval: $[0, 2\pi]$.

Step 1: Check Continuity

- $\cos x$ is continuous everywhere, hence continuous on $[0, 2\pi]$.

Step 2: Check Differentiability

- $\cos x$ is differentiable everywhere; in particular, on $(0, 2\pi)$.

Step 3: Check $f(a) = f(b)$

- $f(0) = \cos 0 = 1$,
- $f(2\pi) = \cos 2\pi = 1$.

- Since $f(0) = f(2\pi)$, the third hypothesis is satisfied.

Conclusion:

All three hypotheses are satisfied; therefore, Rolle's Theorem applies, and there exists at least one $c \in (0, 2\pi)$ such that $f'(c) = 0$.

Additional Considerations and Common Pitfalls

While verifying the hypotheses is straightforward in many cases, some common issues can arise:

1. **Discontinuities within the interval:** Be cautious of functions that are continuous on (a, b) but not on $[a, b]$, especially at the endpoints.

2. **Endpoints where $f(a) \neq f(b)$:** If the function values differ, Rolle's Theorem does not apply; consider whether the Mean Value Theorem might be applicable instead.

3. **Non-differentiable points inside (a, b):** Functions with sharp corners or cusps violate the differentiability hypothesis.

4. **Functions defined piecewise:** Verify continuity and differentiability at the junction points carefully.

Summary of Steps to Verify Rolle's Theorem Hypotheses

Step	Action	Key Points
1	Verify continuity on $[a, b]$	Use function's form and check for discontinuities at endpoints and within the interval.
2	Verify differentiability on (a, b)	Ensure no sharp corners, cusps, or vertical tangents inside the interval.
3	Check if $f(a) = f(b)$	Calculate and compare the endpoint values.

Conclusion

Verifying that a function satisfies the three hypotheses of Rolle's Theorem is an essential step in analyzing the function's behavior within a specified interval. By systematically checking continuity, differentiability, and equal endpoint values, you can confidently determine whether the theorem applies. Once verified, Rolle's Theorem guarantees the existence of at least one point within the interval where the derivative is zero, providing insight into the function's critical points and overall behavior.

Remember, careful verification prevents misapplication of the theorem and ensures accurate mathematical reasoning. Whether you're studying calculus, solving problems, or preparing for exams, mastering this verification process is a valuable skill that enhances your understanding of differential calculus principles.

Ready to practice? Consider various functions and intervals, apply these steps, and see firsthand how Rolle's Theorem helps identify critical points and understand the behavior of functions across different contexts.

Frequently Asked Questions

What are the three hypotheses of Rolle's Theorem that need to be verified for a function on a given interval?

The three hypotheses are: (1) The function is continuous on the closed interval [a, b], (2) The function is differentiable on the open interval (a, b), and (3) The function values at the endpoints are equal, i.e., f(a) = f(b).

How can I verify if a function is continuous on a given interval when applying Rolle's Theorem?

To verify continuity, check if the function is continuous at every point within the interval [a, b], including the endpoints. This can involve analyzing the function's definition, checking for any discontinuities, jumps, or holes within the interval.

What steps should I follow to confirm the differentiability of a function on an interval?

First, ensure the function is differentiable at every point inside the interval (a, b). This involves verifying that the derivative exists at each point, which often requires differentiating the function and checking for any points of non-differentiability such as sharp corners or cusps.

How do I check if the function values at the endpoints are equal in the context of Rolle's Theorem?

Evaluate the function at both endpoints, f(a) and f(b), and compare their values. If f(a) = f(b), then the third hypothesis is satisfied, which is necessary for Rolle's Theorem to hold.

Can you give an example of verifying all three hypotheses of Rolle's Theorem for a specific function?

Certainly! For f(x) = x^2 on [0, 0], the function is continuous and differentiable everywhere, and f(0) = 0 at both endpoints, satisfying all three hypotheses. For a non-trivial example, check f(x) = cos(x) on [0, 2π]; it is continuous, differentiable, and f(0) = f(2π) = 1, so all hypotheses are satisfied.

What should I conclude if I verify all three hypotheses of Rolle's Theorem for a function on an interval?

If all three hypotheses are satisfied, then Rolle's Theorem guarantees that there exists at least one point c in (a, b) where the derivative f'(c) = 0, meaning the function has a horizontal tangent somewhere within the interval.

Additional Resources

Verify That The Function Satisfies The Three Hypotheses Of Rolle's Theorem On The Given Interval is a fundamental step in understanding the behavior of functions within calculus, especially when analyzing the existence of stationary points. Rolle's Theorem is a cornerstone in differential calculus, providing a link between the properties of a function over an interval and the existence of a critical point where the derivative is zero. Before applying the theorem, it is essential to verify that the function in question satisfies all three hypotheses. This ensures the theorem's conclusions are valid and can be confidently used to analyze the function's behavior.

Understanding Rolle's Theorem

Rolle's Theorem states that if a function \(f\) satisfies certain conditions on a closed interval \([a, b]\), then there exists at least one point \(c\) in \((a, b)\) where the derivative \(f'(c)\) is zero. This theorem is often used as a stepping stone for more advanced results like the Mean Value Theorem.

Formal Statement of Rolle's Theorem

Let \(f\) be a function satisfying the following three hypotheses on the interval \([a, b]\):

1. Continuity on \([a, b]\):
\[f \text{ is continuous on } [a, b]\]
2. Differentiability on \((a, b)\):
\[f \text{ is differentiable on } (a, b)\]
3. Equal function values at the endpoints:
\[f(a) = f(b)\]

If these conditions are met, then there exists at least one \(c \in (a, b)\) such that:

\[
f'(c) = 0
\]

Why Verifying The Hypotheses Matters

Before applying Rolle's Theorem, verifying these three hypotheses is crucial because:

- It guarantees the applicability of the theorem.

Without satisfying the hypotheses, the conclusion about the existence of a critical point cannot be assured.

- It prevents incorrect conclusions.
Assuming the theorem's conclusion without verification can lead to errors, especially in complex functions.

- It provides a deeper understanding of the function's behavior.
Verifying hypotheses often clarifies the function's properties over the interval.

Step-by-Step Guide to Verify the Hypotheses

1. Verify Continuity on $[a, b]$

Why?
Continuity ensures the function doesn't have any jumps, breaks, or removable discontinuities on the closed interval, which is essential for applying Rolle's Theorem.

How?
- Identify the function's domain and determine if it is continuous everywhere on $[a, b]$.
- Check for points of discontinuity within the interval.
- Use known continuity rules:

- Polynomial functions are continuous everywhere.
- Rational functions are continuous where the denominator is not zero.
- Trigonometric, exponential, and logarithmic functions have known domains and continuity properties.

Example:
Suppose $f(x) = \sqrt{x}$ on $[0, 4]$.
- Is f continuous on $[0, 4]$?
- Since $\sqrt{x}$ is continuous on $[0, \infty)$, yes, it is continuous there.

2. Verify Differentiability on (a, b)

Why?
Differentiability guarantees the existence of the derivative at every point within the open interval, which is necessary for the conclusion.

How?
- Check the derivative's existence at all points in (a, b).
- Identify points where the derivative may not exist, such as corners, cusps, or vertical tangent lines.
- Use known derivative rules and the function's form.

Example:
For $f(x) = |x|$ on $[-1, 1]$:

- Is f differentiable on $(-1, 1)$?
- At $x=0$, f has a sharp corner, so f is not differentiable there.

- Therefore, Rolle's Theorem does not apply here since the differentiability condition fails at $x=0$.

3. Verify that $f(a) = f(b)$

Why?
The function must take the same value at the endpoints of the interval for the conclusion about a stationary point within (a, b) to hold.

How?
- Calculate $f(a)$ and $f(b)$.
- Compare these values.
- If they are equal, the hypothesis is satisfied; if not, Rolle's Theorem does not apply directly.

Example:
If $f(x) = x^3 - 3x$ on $[-2, 2]$:

- $f(-2) = (-2)^3 - 3(-2) = -8 + 6 = -2$
- $f(2) = 8 - 6 = 2$
- Since $f(-2) \neq f(2)$, the third hypothesis fails, and the theorem does not apply.

Practical Example: Applying the Verification Process

Let's consider a specific function and interval:

Function: $f(x) = x^2 - 4x + 3$
Interval: $[1, 3]$

Step 1: Check Continuity

- $f(x)$ is a polynomial, which is continuous everywhere, including $[1, 3]$.
- Conclusion: Hypothesis 1 satisfied.

Step 2: Check Differentiability

- Polynomials are differentiable everywhere, so $f(x)$ is differentiable on $(1, 3)$.
- Conclusion: Hypothesis 2 satisfied.

Step 3: Check Endpoint Values

- $f(1) = 1 - 4 + 3 = 0$
- $f(3) = 9 - 12 + 3 = 0$
- Since $f(1) = f(3) = 0$, hypothesis 3 satisfied.

Result: All three hypotheses are satisfied, so Rolle's Theorem applies.

Step 4: Find $c \in (1, 3)$ such that $f'(c) = 0$

- $f'(x) = 2x - 4$
- Set $f'(c) = 0$:

$\Rightarrow 2c - 4 = 0$
$\Rightarrow c = 2$

- Check c in $(1, 3)$: Yes, $c=2$ is within the interval.

Conclusion:
By verifying the hypotheses, we've confirmed the existence of a point $c=2$ in $(1, 3)$ where the derivative is zero, consistent with Rolle's Theorem.

Common Pitfalls and Tips

- Neglecting endpoint values: Always verify $f(a) = f(b)$. Even if the function is continuous and differentiable, unequal endpoint values disqualify the theorem's application.

- Overlooking discontinuities: A function might be continuous but not differentiable at certain points, invalidating the hypothesis.

- Misjudging differentiability: Remember that functions with sharp corners, cusps, or vertical tangents are not differentiable at those points.

- Interval considerations: Ensure the interval is closed $[a, b]$, as Rolle’s Theorem requires.

Extending Beyond Rolle's Theorem

Once the hypotheses are verified, and the theorem's conclusion is established, this can lead to further analysis, such as:

- Identifying critical points in the context of optimization problems.
- Applying the Mean Value Theorem for functions satisfying similar hypotheses but with $f(a) \neq f(b)$.
- Analyzing function behavior to understand monotonicity and concavity.

Final Thoughts

Verifying that the function satisfies the three hypotheses of Rolle's Theorem on the given interval is an essential skill in calculus that underpins many theoretical and practical applications. It ensures the rigorous application of the theorem and lays the groundwork for deeper insights into the behavior of functions. Always approach this verification systematically, checking continuity, differentiability, and endpoint values, and remember that each hypothesis plays a vital role in guaranteeing the existence of stationary points within the interval. With practice, this process becomes intuitive and forms a foundational part of your calculus toolkit.

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