Write The Function In Terms Of Unit Step Functions. Find The Laplace Transform Of The Given Function.

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Understanding how to express functions using unit step functions (also known as Heaviside functions) and then finding their Laplace transforms is a fundamental skill in engineering and mathematics, especially in the analysis of systems with switching behaviors or piecewise-defined inputs. This article provides a comprehensive guide to rewriting functions with unit step functions and computing their Laplace transforms step-by-step, ensuring clarity for learners at all levels.

Introduction to Unit Step Functions and Laplace Transforms

Before diving into specific examples, it's essential to understand the core concepts involved—namely, the unit step function and Laplace transform.

What Is a Unit Step Function?

The unit step function, denoted as $u(t - a)$, is a mathematical tool used to model sudden changes or switching behavior in functions. It is defined as:

$$
u(t - a) = \begin{cases}
0, & t < a \\
1, & t \geq a
\end{cases}
$$

This function effectively "turns on" a particular part of a function at the point $t = a$. By combining multiple unit step functions, complex piecewise functions can be represented in a unified expression.

What Is the Laplace Transform?

The Laplace transform is an integral transform used to convert functions from the time domain into the complex frequency domain. It simplifies differential equations into algebraic equations, making analysis and solutions more straightforward.

The Laplace transform of a function $f(t)$ (defined for $t \geq 0$) is given by:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt
$$

where s is a complex variable, $s = \sigma + j\omega$.

Expressing Functions in Terms of Unit Step Functions

Many functions encountered in engineering are piecewise, such as signals that switch on or off at specific times. To analyze these systematically, we rewrite them using unit step functions.

Why Use Unit Step Functions?

- To simplify the handling of piecewise functions.
- To facilitate the use of Laplace transforms on functions that are not defined over the entire domain uniformly.
- To enable algebraic manipulation of switching behaviors.

Methodology for Rewriting Functions

Suppose a piecewise function:

$$
f(t) = \begin{cases}
f_1(t), & a \leq t < b \\
f_2(t), & t \geq b
\end{cases}
$$

This can be expressed as:

$$
f(t) = f_1(t) \cdot u(t - a) + [f_2(t) - f_1(t)] \cdot u(t - b)
$$

This representation "activates" different parts of the function at specified points using unit step functions.

Step-by-Step Procedure to Write Functions in Terms of Unit Step Functions

1. Identify the intervals where different expressions are valid.

2. Express each segment as a function multiplied by an appropriate unit step function that turns it on at the correct time.

3. Combine all parts into a single expression.

4. Simplify the expression where possible.

Example: Piecewise Function Conversion

Suppose you have the function:

$$
f(t) =
\begin{cases}
0, & t < 2 \\
t - 2, & t \geq 2
\end{cases}
$$

Expressed using the unit step function:

$$
f(t) = (t - 2) \cdot u(t - 2)
$$

Here, for $t < 2$, $u(t - 2) = 0$, so $f(t) = 0$, and for $t \geq 2$, $u(t - 2) = 1$, so $f(t) = t - 2$.

Computing the Laplace Transform of the Function

Once the function is expressed in terms of unit step functions, computing its Laplace transform involves applying the linearity of the transform and known transforms of basic functions.

Key Properties of Laplace Transforms

- Linearity:
$$
\mathcal{L}\{a f(t) + b g(t)\} = a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\}
$$
- Transform of a shifted function:
$$
\mathcal{L}\{f(t - a) u(t - a)\} = e^{-a s} \mathcal{L}\{f(t)\}
$$
- Transform of the unit step function:
$$
\mathcal{L}\{u(t - a)\} = \frac{e^{-a s}}{s}
$$

Using these properties, the process becomes systematic.

Step-by-Step Calculation: An Example

Let's work through an example to illustrate the process thoroughly.

Given function:
$$
f(t) =
\begin{cases}
0, & t < 1 \\
$$

$$
2(t - 1), & t \geq 1
\end{cases}
$$

Step 1: Rewrite using unit step functions:

$$
f(t) = 2(t - 1) u(t - 1)
$$

Step 2: Find the Laplace transform:

$$
F(s) = \mathcal{L}\{2(t - 1) u(t - 1)\}
$$

Using the property:

$$
\mathcal{L}\{f(t - a) u(t - a)\} = e^{-a s} \mathcal{L}\{f(t)\}
$$

Let $f(t) = 2t$, then:

$$
F(s) = e^{-1 \cdot s} \cdot \mathcal{L}\{2 t\}
$$

The Laplace transform of $2 t$:

$$
\mathcal{L}\{2 t\} = 2 \times \frac{1}{s^2} = \frac{2}{s^2}
$$

Therefore,

$$
F(s) = e^{-s} \times \frac{2}{s^2} = \frac{2 e^{-s}}{s^2}
$$

Result:

$$
\boxed{
F(s) = \frac{2 e^{-s}}{s^2}
}
$$

Generalization to More Complex Functions

The methodology extends to more complex piecewise functions, including multiple intervals and different functions on each interval.

Example: Multiple Switches

Suppose:

$$
f(t) =
\begin{cases}
0, \text{ if } t < 1 \\
t, \text{ if } 1 \leq t < 3 \\
2t, \text{ if } t \geq 3
\end{cases}
$$

Expressed as:

$$
f(t) = t \cdot u(t - 1) + [2t - t] \cdot u(t - 3) = t \cdot u(t - 1) + t \cdot u(t - 3)
$$

But note that for $t \geq 3$, both terms are active, so to avoid double counting, the function can be written as:

$$
f(t) = t \cdot u(t - 1) + t \cdot u(t - 3) - t \cdot u(t - 3)
$$
which simplifies to the original piecewise form.

Computing the Laplace transform:

$$
F(s) = \mathcal{L}\{t u(t - 1)\} + \mathcal{L}\{t u(t - 3)\}
$$

Using the shifting property:

$$
\mathcal{L}\{t u(t - a)\} = e^{-a s} \times \mathcal{L}\{\text{shifted } t\}
$$

But since t is not shifted, more appropriately, note that:

$$
\mathcal{L}\{t u(t - a)\} = e^{-a s} \times \left(\frac{1}{s^2} \right) + \text{additional terms}
$$

Alternatively, it's often easier to write $t u(t - a)$ as:

$$
t u(t - a) = (t) u(t - a)
$$

And then, using the property:

$$
\mathcal{L}\{f(t - a) u(t - a)\} = e^{-a s} \mathcal{L}\{f(t)\}
$$

which implies that:

$$
\mathcal{L}\{t u(t - a)\} = e^{-a s} \left(\frac{1}{s^2} \right) + a e^{-a s} \times \frac{1}{s}
$$

\]

Frequently Asked Questions

How can a piecewise function be expressed using unit step functions?

A piecewise function can be expressed as a sum of functions multiplied by unit step functions to turn each piece on or off at specified points. For example, $f(t) = f_1(t)u(t - a) + f_2(t)u(t - b) - f_1(t)u(t - b)$, where $u(t - c)$ is the unit step function shifting the activation point.

What is the general approach to find the Laplace Transform of a function written in terms of unit step functions?

The general approach involves expressing the piecewise function using unit step functions, then applying the Laplace Transform properties, such as linearity and the shifting theorem, to evaluate the transform of each term individually.

How does the Laplace Transform handle functions involving unit step functions?

The Laplace Transform of a unit step function $u(t - a)$ multiplied by a function $f(t)$ is e^{-as} times the Laplace Transform of $f(t + a)$, allowing for straightforward evaluation of piecewise-defined functions.

Can you illustrate the process of transforming a simple piecewise function into a Laplace Transform using unit step functions?

Yes, for example, the function $f(t) = 0$ for $t < a$, and $f(t) = g(t)$ for $t \geq a$ can be written as $f(t) = g(t)u(t - a)$. Its Laplace Transform is $L\{f(t)\} = e^{-as} L\{g(t + a)\}$, enabling easier computation.

What are the key benefits of representing functions with unit step functions before applying the Laplace Transform?

Representing functions with unit step functions simplifies handling discontinuities and piecewise definitions, making the application of Laplace Transform straightforward by leveraging properties like linearity and time shifting.

Additional Resources

Unit Step Functions and Laplace Transform: An In-Depth Exploration

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Introduction: Demystifying the Unit Step Function and Its Significance

In the realm of engineering, mathematics, and signal processing, the analysis of complex functions often necessitates breaking down intricate signals into simpler, manageable components. Among these tools, the unit step function — also known as the Heaviside function — stands out as a fundamental building block. It enables us to model signals that switch on at a specific point in time, such as electronic circuits turning on, mechanical systems starting operation, or control systems activating.

Simultaneously, the Laplace transform is a powerful mathematical technique that converts functions from the time domain into the complex frequency domain. This transformation simplifies differential equations, making them more straightforward to analyze and solve. When combined with the unit step function, it allows for elegant handling of piecewise or discontinuous signals.

This article aims to provide an in-depth understanding of how to express functions using the unit step functions and compute their Laplace transforms. Whether you're an engineering student, a professional in control systems, or someone eager to deepen your mathematical toolset, this guide aims to clarify these concepts through comprehensive explanations, detailed examples, and practical insights.

Part 1: Understanding the Unit Step Function

What Is the Unit Step Function?

The unit step function, denoted as $u(t)$, is mathematically defined as:

$$
u(t) = \begin{cases}
0, & t < 0 \\
1, & t \geq 0
\end{cases}
$$

It acts like a switch that "turns on" at $t = 0$. This simple yet powerful function allows us to model signals that start at a specific moment, which is crucial when dealing with real-world systems that are activated at certain times.

Key characteristics:

- It is zero for negative time values.
- It switches to one at $t = 0$ and remains there.
- It can be shifted in time to model activation at different moments.

Mathematical Representation and Properties

The unit step function can be used to construct more complex piecewise functions. For example, a function that activates at $t = a$ can be represented as:

$$
f(t) = f(t) \cdot u(t - a)
$$

This notation indicates that $f(t)$ is "turned on" at $t = a$.

Properties include:

- Linearity: $a \cdot u(t) + b \cdot u(t) = (a + b) \cdot u(t)$

- Shifting: $u(t - a)$ shifts the step to activate at $t = a$
- Scaling: The step function can be scaled by constants to model varying activation intensities

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Expressing Piecewise Functions Using Unit Step Functions

One of the main advantages of the unit step function is its ability to express piecewise functions in a single, unified expression. For example, consider the following piecewise function:

$$
f(t) = \begin{cases}
0, & t < 0 \\
t, & 0 \leq t < 2 \\
3, & t \geq 2
\end{cases}
$$

This can be rewritten using unit step functions as:

$$
f(t) = t \cdot u(t) + (3 - t) \cdot u(t - 2)
$$

In this form:

- $t \cdot u(t)$ models the linear increase from 0 onward.
- $(3 - t) \cdot u(t - 2)$ models the change at $t = 2$.

This approach simplifies the process of taking transforms, derivatives, or integrals of such functions.

Part 2: Expressing Functions in Terms of Unit Step Functions

Step-by-Step Methodology

Transforming a piecewise function into an expression involving unit step functions involves:

1. Identify the intervals and corresponding function values.
2. Express each segment as a function multiplied by step functions that activate at the appropriate points.
3. Sum all terms to obtain the complete expression.

Example:

Given the piecewise function:

$$
f(t) = \begin{cases}
0, & t < 0 \\
t, & 0 \leq t < 3 \\
5, & t \geq 3
\end{cases}

\]

Expressed with unit step functions:

$$
f(t) = t \cdot u(t) + (5 - t) \cdot u(t - 3)
$$

Here, $t \cdot u(t)$ models the linear increase starting at 0, while $(5 - t) \cdot u(t - 3)$ "activates" at $t = 3$, adjusting the function to a constant value.

General Formula for Piecewise Functions

For a general piecewise function:

$$
f(t) = \begin{cases}
f_1(t), & t < a_1 \\
f_2(t), & a_1 \leq t < a_2 \\
\vdots \\
f_n(t), & t \geq a_{n-1}
\end{cases}
$$

The expression in terms of unit step functions is:

$$
f(t) = f_1(t) + \left[f_2(t) - f_1(t) \right] u(t - a_1) + \left[f_3(t) - f_2(t) \right] u(t - a_2) + \cdots + \left[f_n(t) - f_{n-1}(t) \right] u(t - a_{n-1})
$$

This method ensures a concise, algebraic representation suitable for transform calculations.

Part 3: Laplace Transform of Functions Expressed with Unit Step Functions

Introduction to the Laplace Transform

The Laplace transform of a function $f(t)$, denoted $\mathcal{L}\{f(t)\}$, is defined as:

$$
\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \, dt
$$

where:

- s is a complex frequency variable $s = \sigma + j\omega$
- The integral converges for functions of exponential order

This transformation converts differential equations into algebraic equations, significantly simplifying analysis.

Laplace Transform of the Unit Step Function

The Laplace transform of the basic unit step function $u(t - a)$ is:

$$
\mathcal{L}\{ u(t - a) \} = \frac{e^{-a s}}{s}
$$

This exponential factor accounts for the shift in activation time.

Key properties:

- The Laplace transform of a shifted function $f(t - a) u(t - a)$ is $e^{-a s} \times \mathcal{L}\{f(t)\}$

Transforming Piecewise Functions Expressed in Terms of Unit Step Functions

When a function is written as a sum involving unit step functions, the Laplace transform can be computed term-by-term, leveraging linearity:

$$
\mathcal{L}\{f(t)\} = \sum_{i} \mathcal{L}\{f_i(t) \cdot u(t - a_i)\}
$$

Using the property:

$$
\mathcal{L}\{f(t - a) u(t - a)\} = e^{-a s} \times \mathcal{L}\{f(t + a)\}
$$

or, if $f(t)$ is a simple function like t, e^{kt}, etc., standard Laplace transforms can be applied directly, with the exponential shift incorporated.

Example:

Suppose:

$$
f(t) = t \cdot u(t) + (5 - t) \cdot u(t - 3)
$$

The Laplace transform becomes:

$$
\mathcal{L}\{f(t)\} = \mathcal{L}\{t \cdot u(t)\} + \mathcal{L}\{ (5 - t) \cdot u(t - 3) \}
$$

Calculating:

- $\mathcal{L}\{t\} = \frac{1}{s^2}$

- For $(5 - t) \cdot u(t - 3)$:

$$
\Rightarrow e^{-3s} \times \mathcal{L}\{ (5 - (t + 3)) \} \text{, with } t + 3 \text{ substitution}
$$

which simplifies further to standard forms.

Part 4: Practical Applications and Examples

Modeling Real-World Signals

Using unit step functions combined with the Laplace transform allows engineers and scientists to analyze:

- Circuit responses to sudden

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