3. (3 Pts) Express U(105) As An External Direct Product In Three Ways. There Is A Fact We Mentioned In

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Understanding the structure of groups is a fundamental aspect of group theory in abstract algebra. One of the essential tools for analyzing complex groups is expressing them as external direct products of simpler groups. In particular, the group $U(105)$—the group of units modulo 105—serves as an excellent example to illustrate this concept. This article explores three distinct methods to express $U(105)$ as an external direct product, shedding light on the underlying algebraic structures and key facts that facilitate these decompositions.

Introduction to $U(105)$: The Group of Units Modulo 105

Before delving into the methods of expressing $U(105)$ as an external direct product, it is crucial to understand what this group represents.

Definition:
$U(n)$ denotes the group of units modulo n, comprising all integers less than n that are coprime to n, equipped with multiplication modulo n.

Properties of $U(105)$:
- Since $105 = 3 \times 5 \times 7$, the structure of $U(105)$ can be explored using properties of cyclic groups and the Chinese Remainder Theorem (CRT).
- The order of $U(105)$ is given by Euler's totient function:
$$|U(105)| = \varphi(105) = \varphi(3) \times \varphi(5) \times \varphi(7) = 2 \times 4 \times 6 = 48.$$
- $U(105)$ is a finite abelian group of order 48, and its structure can be decomposed into simpler components.

Method 1: Using the Chinese Remainder Theorem (CRT)

The first way to express $U(105)$ as an external direct product is through the application of the Chinese Remainder Theorem, which states that for pairwise coprime integers n_1, n_2, n_3, the following is an isomorphism:

$$
U(n_1 n_2 n_3) \cong U(n_1) \times U(n_2) \times U(n_3)
$$

Step-by-step process:

1. Factorization of 105:
Since $105 = 3 \times 5 \times 7$, the prime factors are pairwise coprime.

2. Applying the CRT to the multiplicative groups:
The isomorphism holds as:

$$
U(105) \cong U(3) \times U(5) \times U(7)
$$

3. Structure of each component:

- $U(3)$: Since 3 is prime, $U(3)$ has $\varphi(3) = 2$ elements, isomorphic to C_2.
- $U(5)$: $\varphi(5) = 4$, isomorphic to C_4.
- $U(7)$: $\varphi(7) = 6$, isomorphic to C_6.

4. Resulting decomposition:

$$
U(105) \cong C_2 \times C_4 \times C_6
$$

Summary of Method 1:
By applying the Chinese Remainder Theorem, $U(105)$ can be expressed as the external direct product of the units modulo its prime factors:

$$
\boxed{
U(105) \cong U(3) \times U(5) \times U(7) \cong C_2 \times C_4 \times C_6
}
$$

Method 2: Decomposition into Cyclic Subgroups via the Fundamental Theorem of Finite Abelian Groups

The second method leverages the fundamental theorem of finite abelian groups, which states that every finite abelian group is a direct product of cyclic groups of prime power order.

Key steps:

1. Identify the structure of each component group:

- $U(3) \cong C_2$: Since 3 is prime, $U(3)$ is cyclic of order 2.

- $U(5) \cong C_4$: Because $\varphi(5) = 4$, and the group is cyclic.

- $U(7) \cong C_6$: With $\varphi(7) = 6$, the group is cyclic.

2. Expressing $U(105)$ as a product of cyclic groups:

$$
U(105) \cong C_2 \times C_4 \times C_6
$$

3. Further decomposition into elementary factors:

- Recognize that C_6 can be written as $C_2 \times C_3$ due to the Chinese Remainder Theorem for cyclic groups of order 6.

- Thus,

$$
U(105) \cong C_2 \times C_4 \times C_2 \times C_3
$$

4. Final decomposition:

$$
\boxed{
U(105) \cong C_2 \times C_2 \times C_3 \times C_4
}
$$

This decomposition highlights the internal structure of $U(105)$, illustrating it as a product of cyclic groups of prime power order, with some factors possibly combined or further broken down.

Method 3: Explicit Construction of the External Direct Product via Generators and Relations

The third approach involves explicitly constructing $U(105)$ as an external direct product by identifying generators and their relations, then expressing the group as a product of smaller, well-understood groups.

Step-by-step process:

1. Identify generators of $U(105)$:

- Find elements of $U(105)$ that generate the entire group or significant subgroups.

- Use known generators of $U(3)$, $U(5)$, and $U(7)$. For example:
- $U(3) = \{1, 2\}$, with generator 2.
- $U(5) = \{1, 2, 3, 4\}$, with generator 2.
- $U(7) = \{1, 2, 3, 4, 5, 6\}$, with generator 3.

2. Construct the group as a product:

- Map elements of $U(105)$ to their images in the respective $U(p)$ groups, respecting their generator relations.

3. Define the group operations explicitly:

- For each component, specify the relations among generators.

- For example, in C_2, the generator squared is the identity; in C_4, the generator raised to the 4th power is identity, etc.

4. Express $U(105)$ as an external direct product:

$$
U(105) \cong \langle g_3 \rangle \times \langle g_5 \rangle \times \langle g_7 \rangle
$$

with relations:

$$
g_3^2 = e, \quad g_5^4 = e, \quad g_7^6 = e
$$

Summary of Method 3:
By explicitly constructing generators and relations, $U(105)$ can be viewed as an external direct product of cyclic groups:

$$
\boxed{
U(105) \cong C_2 \times C_4 \times C_6
}
$$

where the individual cyclic groups correspond to the multiplicative units modulo 3, 5, and 7, respectively.

Key Fact: The Fundamental Theorem of Finite Abelian Groups

A central fact that underpins all three methods is the Fundamental Theorem of Finite Abelian Groups, which states:

> Every finite abelian group is isomorphic to a direct product of cyclic groups of prime power order, uniquely up to the order of the factors.

This theorem allows algebraists to classify and decompose groups like \(U(105) \) systematically. It guarantees that \(U(105) \), being finite and abelian, can be expressed as a product involving cyclic groups of prime power order, which can then be further analyzed or constructed explicitly.

Conclusion: The Versatility of Expressing \(U(105) \) as an External Direct Product

Expressing the group \(U(105) \) as an external direct product reveals its internal structure, simplifies its analysis, and provides insights into its properties such as cyclicity, order, and subgroup composition. The three

Frequently Asked Questions

What does expressing U(105) as an external direct product involve in algebraic structures?

It involves decomposing the group U(105), the multiplicative group of units modulo 105, into a direct product of smaller, simpler groups, typically cyclic groups, based on the prime factorization of 105.

How can U(105) be expressed as an external direct product in three different ways?

U(105) can be expressed as a direct product of groups such as U(3) x U(5) x U(7), or alternatively as combinations like U(3) x U(35), or U(15) x U(7), depending on how the factors are chosen based on the prime factors of 105.

What is the significance of the fact mentioned in expressing U(105) as an external direct product?

The fact refers to the Chinese Remainder Theorem, which states that the structure of U(105) can be decomposed into a product of groups corresponding to its prime factors, facilitating multiple ways of expressing it as an external direct product.

What are the prime factors of 105, and how do they influence the decomposition of U(105)?

The prime factors of 105 are 3, 5, and 7. These factors allow us to decompose U(105) into direct products involving U(3), U(5), and U(7), or their combinations, based on the Chinese Remainder Theorem.

Can you provide an example of expressing U(105) as an external direct product in three different ways?

Yes. For example: 1) U(105) ≅ U(3) x U(5) x U(7); 2) U(105) ≅ U(15) x U(7); 3) U(105) ≅ U(3) x U(35). Each decomposition reflects different subgroup combinations based on divisors of 105.

What is the order of the groups involved when expressing U(105) as an external direct product?

The order of U(105) is φ(105) = 48, which factors into the orders of the component groups in the direct product, such as φ(3)=2, φ(5)=4, and φ(7)=6, which multiply accordingly to give the total.

Why is understanding the external direct product of U(105) important in group theory?

It helps in analyzing the structure of the group, simplifies computations, and provides insights into its subgroup composition, automorphisms, and applications in number theory and cryptography.

What is the role of the Chinese Remainder Theorem in expressing U(105) as a direct product?

The Chinese Remainder Theorem guarantees an isomorphism between U(105) and the direct product of units modulo its prime power factors, enabling multiple decompositions into external direct products.

Are the different ways of expressing U(105) as an external direct product always unique?

No, the decompositions depend on how you factor and group the prime components; multiple valid decompositions exist, each highlighting different subgroup structures based on divisors of 105.

How can understanding these multiple expressions of U(105) aid in practical applications?

It aids in simplifying calculations in modular arithmetic, designing cryptographic algorithms, and understanding the structure of multiplicative groups in number theory problems.

Additional Resources

Expressing U(105) as an External Direct Product in Three Ways: An In-Depth Analysis

In the realm of algebra and group theory, understanding the structure of groups such as the unit group modulo an integer is fundamental. One such group, U(105), the group of units modulo 105, holds particular interest due to its rich structure derived from the properties of modular arithmetic and the Chinese Remainder Theorem. This article explores how to express U(105) as an external direct product in three distinct ways, revealing the underlying algebraic architecture. We will delve into the theoretical background, detailed methods, and implications of these decompositions to provide a comprehensive understanding suitable for mathematicians, students, and enthusiasts alike.

Understanding U(105): The Basics

Before we explore the various decompositions, it is essential to understand what U(105) represents.

What is U(105)?

U(105) is the multiplicative group of integers modulo 105 that are coprime to 105. Formally,

$$
U(105) = \{ x \in \mathbb{Z} \mid 1 \leq x \leq 104, \gcd(x, 105) = 1 \}
$$

with the group operation being multiplication modulo 105.

Basic Properties

- Order of U(105): The size of the group, denoted |U(105)|, equals Euler's totient function $\varphi(105)$.

- Calculating $\varphi(105)$: Since 105 factors as $3 \times 5 \times 7$, the Euler totient function is multiplicative over these factors:

$$
\varphi(105) = \varphi(3) \times \varphi(5) \times \varphi(7) = (3 - 1) \times (5 - 1) \times (7 - 1) = 2 \times 4 \times 6 = 48
$$

Thus, |U(105)| = 48.

Structural Significance

The structure of U(105) reflects the interplay of its prime factors and the Chinese Remainder Theorem (CRT), which allows the decomposition of the group into direct products of groups modulo prime powers.

Decomposition of U(105): Theoretical Foundations

The core idea behind expressing U(105) as an external direct product lies in the CRT and the properties of cyclic groups.

Chinese Remainder Theorem (CRT)

CRT states that for pairwise coprime integers $n_1, n_2, \ldots, n_k$:

$$
\mathbb{Z} / (n_1 n_2 \cdots n_k) \cong \mathbb{Z} / n_1 \times \mathbb{Z} / n_2 \times \cdots \times \mathbb{Z} / n_k
$$

This extends to the unit groups:

$$
U(n_1 n_2 \cdots n_k) \cong U(n_1) \times U(n_2) \times \cdots \times U(n_k)
$$

when the n_i are pairwise coprime, which is the case for 3, 5, and 7.

Applying the CRT to U(105)

Since 105 factors as the product of 3, 5, and 7:

$$
U(105) \cong U(3) \times U(5) \times U(7)
$$

This gives us the first natural way to express U(105) as an external direct product.

Additional Structural Decompositions

Beyond the straightforward CRT decomposition, we can explore alternative decompositions based on the internal structure of the groups $U(n)$, especially their cyclicity, and the nature of their subgroups.

Method 1: Direct Application of CRT — The Canonical Decomposition

The most straightforward approach to express U(105) as an external direct product is by directly applying the Chinese Remainder Theorem, which leverages the coprimality of 3, 5, and 7.

Step-by-Step Process

1. Factorization of 105:

$$
105 = 3 \times 5 \times 7
$$

2. Identify the unit groups modulo each prime:

- $U(3) \cong \mathbb{Z}_2$, since $\varphi(3) = 2$, and the units modulo 3 are $\{1, 2\}$.

- $U(5) \cong \mathbb{Z}_4$, since $\varphi(5) = 4$, units are $\{1, 2, 3, 4\}$.

- $U(7) \cong \mathbb{Z}_6$, since $\varphi(7) = 6$, units are $\{1, 2, 3, 4, 5, 6\}$.

3. Express U(105) as a direct product:

$$
U(105) \cong U(3) \times U(5) \times U(7)
$$

which simplifies to:

$$
U(105) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6
$$

Significance and Implications

This decomposition reveals that U(105) is isomorphic to a product of cyclic groups, with orders 2, 4, and 6, respectively. The group structure can be further analyzed by examining the direct product of these cyclic groups, which is crucial in understanding elements, subgroups, and automorphisms within U(105).

Method 2: Decomposition into Cyclic Components — Using the Structure of U(p^k)

While the first method relies on the CRT, the second approach involves decomposing each $U(p^k)$ into cyclic components more granularly, especially for prime powers.

Analyzing $U(3)$, $U(5)$, and $U(7)$

- $U(3)$:

Since 3 is prime, $U(3) \cong \mathbb{Z}_2$, cyclic of order 2.

- $U(5)$:

5 is prime, so $U(5) \cong \mathbb{Z}_4$, cyclic of order 4.

- $U(7)$:

7 is prime, so $U(7) \cong \mathbb{Z}_6$, cyclic of order 6.

Decomposition into Cyclic Groups

- For $U(3)$:

Already cyclic, isomorphic to $\mathbb{Z}_2$.

- For $U(5)$:

Cyclic, isomorphic to $\mathbb{Z}_4$. The generator can be 2, since 2 is primitive modulo 5.

- For $U(7)$:

Cyclic, isomorphic to $\mathbb{Z}_6$. For example, 3 is a primitive root modulo 7.

Combining the Components

Expressing U(105) as a direct product of these cyclic groups:

$$
U(105) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6
$$

This explicit decomposition into cyclic groups provides insights into the group's automorphism structure, element orders, and subgroup lattice.

Additional Observations

- The overall structure is a product of cyclic groups of orders 2, 4, and 6.

- The total order:

$$
2 \times 4 \times 6 = 48
$$

matching the order $\varphi(105)$.

- The decomposition is not necessarily cyclic because the direct product of cyclic groups need not be cyclic unless their orders are coprime. Since 2, 4, and 6 are not pairwise coprime, the group U(105) is not cyclic but a direct product of cyclic groups with shared factors.

Method 3: Alternative Decomposition via Subgroup Structures and Isomorphism Classes

Beyond the explicit factorization via the CRT and cyclic group breakdowns, an alternative way to express U(105) involves analyzing its subgroup structure and isomorphism classes to find different decompositions.

Recognizing the Decomposition into a Product of Smaller Groups

Given that:

$$
U(105) \cong U(3) \times U(5) \times U(7)
$$

and each component is known, we can consider further decompositions:

- Express $U(3)$ as $\mathbb{Z}_2$:

Since $U(3)$ is cyclic of order 2, trivial to express.

- Express $U(5)$ as $\mathbb{Z}_4$:

Also cyclic, generated by 2 modulo 5.

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