A Certain Infinite Geometric Series Has First Term 7 And Sum 4. What Is The Result When The Third Term

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Understanding the properties of geometric series is fundamental in mathematics, especially when dealing with infinite sums, sequences, and series. In this comprehensive article, we will explore a specific problem involving an infinite geometric series with a first term of 7 and a sum of 4, and then determine what the third term of this series is. This problem not only tests your understanding of geometric series but also enhances your problem-solving skills in algebra and series analysis.

Introduction to Geometric Series

Before diving into the problem, it's essential to review the key concepts related to geometric series.

What Is a Geometric Series?

A geometric series is a series of numbers where each term after the first is obtained by multiplying the previous term by a constant called the common ratio (denoted as r). Mathematically, a geometric series can be expressed as:

$$ a + ar + ar^2 + ar^3 + \dots $$

where:
- a is the first term,
- r is the common ratio.

Finite vs. Infinite Geometric Series

- Finite geometric series have a limited number of terms, and their sum can be calculated using the formula:

$$ S_n = a \frac{1 - r^n}{1 - r} $$

where n is the number of terms.

- Infinite geometric series extend indefinitely, and their sum converges only if the absolute value of the common ratio is less than 1 ($|r| < 1$). The sum of an infinite geometric series with $|r| < 1$ is:

$$S_{\infty} = \frac{a}{1 - r}$$

In our problem, since the series is infinite with a finite sum, the condition $|r| < 1$ must hold.

Analyzing the Given Problem

The problem states:

> An infinite geometric series has a first term $a = 7$ and a sum $S_{\infty} = 4$. What is the third term?

This information provides two critical pieces:
1. The first term a is 7.
2. The sum of the infinite series is 4.

Our goal is to find the third term T_3.

Step 1: Find the Common Ratio r

Using the formula for the sum of an infinite geometric series:

$$S_{\infty} = \frac{a}{1 - r}$$

we can substitute the known values:

$$4 = \frac{7}{1 - r}$$

Solving for r:

$$1 - r = \frac{7}{4}$$

$$1 - r = 1.75$$

$$r = 1 - 1.75 = -0.75$$

Result: The common ratio $r = -0.75$.

Important Note: Since $|r| = 0.75 < 1$, the series converges, confirming the validity of the sum formula.

Step 2: Find the Third Term T_3

In a geometric series, the n-th term T_n is given by:

$$T_n = a \times r^{n-1}$$

Applying this to find the third term:

$$T_3 = a \times r^{2} = 7 \times (-0.75)^2$$

Calculating:

$$T_3 = 7 \times 0.5625 = 3.9375$$

Result: The third term $T_3 = 3.9375$.

Deeper Understanding and Additional Insights

Understanding how to manipulate and interpret the properties of geometric series enhances problem-solving skills, especially in more complex scenarios involving series convergence, divergence, and partial sums.

Why Is the Common Ratio Negative?

In our problem, the negative ratio indicates that the series alternates in sign:

- The first term is positive: 7.
- The second term: $7 \times -0.75 = -5.25$.
- The third term: $-5.25 \times -0.75 = 3.9375$.

This alternating pattern is characteristic of series with negative ratios.

Implications of the Negative Ratio

- The terms alternate in sign, which can affect the convergence behavior.
- The sum converges because the absolute value of ratio $|r| = 0.75 < 1$.
- The series approaches the sum of 4, balancing the positive and negative terms over an infinite number of terms.

Verifying the Results

It's always good practice to verify the calculations:

- Check the sum of the series:

$$S_{\infty} = \frac{a}{1 - r} = \frac{7}{1 - (-0.75)} = \frac{7}{1 + 0.75} = \frac{7}{1.75} = 4$$

Confirmed.

- Calculate the third term directly:

$$T_3 = 7 \times (-0.75)^2 = 7 \times 0.5625 = 3.9375$$

Correct.

Practical Applications of Geometric Series

Understanding geometric series has numerous real-world applications:

- **Finance:** Calculating loan amortizations, compound interest, and investment growth.
- **Physics:** Analyzing decay processes or wave amplitudes that decrease geometrically.
- **Computer Science:** Algorithm analysis involving recursive algorithms or divide-and-conquer strategies.

- **Engineering:** Signal processing and control systems often utilize geometric series for system responses.

Summary and Key Takeaways

- The problem involved an infinite geometric series with first term $a = 7$ and sum $S_{\infty} = 4$.
- Using the formula $S_{\infty} = \frac{a}{1 - r}$, we found the common ratio $r = -0.75$.
- The third term of the series is computed using $T_3 = a \times r^{2}$, resulting in 3.9375.
- The negative ratio indicates an alternating series, which converges due to $|r| < 1$.

Conclusion

This detailed exploration demonstrates how to approach a problem involving an infinite geometric series, emphasizing the importance of understanding series formulas, algebraic manipulation, and the behavior of geometric sequences. Recognizing the significance of the common ratio and its impact on series convergence is crucial for solving similar problems in mathematics and related fields.

By mastering these concepts, students and professionals can confidently analyze series in various contexts, from theoretical mathematics to practical applications in science and engineering.

Frequently Asked Questions

What is the common ratio of the infinite geometric series with first term 7 and sum 4?

The common ratio r can be found using the formula for the sum of an infinite geometric series: S = a / (1 - r). Substituting S=4 and a=7, we get 4 = 7 / (1 - r). Solving for r: 1 - r = 7/4, so r = 1 - 7/4 = -3/4.

How do you find the third term of the geometric series given the first term and ratio?

The third term a_3 is found using the formula $a_3 = a\, r^2$. With a=7 and r=-3/4, $a_3 = 7\, (-3/4)^2 = 7\, (9/16) = 63/16$.

What is the value of the third term in the series?

The third term is 63/16, which is approximately 3.9375.

What is the significance of the sum being 4 for this series?

It indicates that the series converges and allows us to determine the common ratio, which in this case is -3/4, ensuring the absolute value is less than 1.

Can the sum of an infinite geometric series be less than the first term? Why or why not?

Yes, if the common ratio is between -1 and 1 (excluding -1 and 1), the series converges, and the sum can be less than the first term depending on the ratio. In this case, the sum is less than the first term because the ratio is negative and less than 1 in magnitude.

Is the series with the given parameters an increasing or decreasing sequence?

Since the common ratio is negative (-3/4), the terms alternate in sign, and the sequence oscillates, decreasing in magnitude towards zero.

Additional Resources

Exploring the Infinite Geometric Series: Unraveling the Mystery of the Third Term

The concept of infinite geometric series is a fundamental topic in mathematics that combines elegance with practical applications. When a series is infinite, yet converges to a finite sum, it opens a window into the fascinating world of mathematical limits and ratios. In this article, we delve into a specific problem: determining the third term of an infinite geometric series given its first term and the sum of the series. This inquiry not only sharpens our understanding of geometric progressions but also illustrates the power of algebraic methods in solving real-world problems.

Understanding Geometric Series: Definitions and Basic Concepts

Before unraveling the specific problem, it is essential to establish a clear understanding of what an infinite geometric series entails.

What Is a Geometric Series?

A geometric series is a sequence of numbers where each term after the first is found by multiplying the previous term by a fixed, non-zero constant called the common ratio (denoted as r). Mathematically, if the first term is a, the series can be expressed as:

$$ a + ar + ar^2 + ar^3 + \ldots $$

The series continues infinitely, and the terms follow the pattern dictated by a and r.

Finite vs. Infinite Series

- Finite geometric series have a limited number of terms. The sum of n terms is given by:

$$ S_n = a \frac{1 - r^n}{1 - r} \quad \text{(for } r \neq 1) $$

- Infinite geometric series extend indefinitely. When does such a series have a finite sum? The critical condition is that the absolute value of the common ratio must be less than 1: $|r| < 1$. Under this condition, the sum of the infinite series is:

$$ S_\infty = \frac{a}{1 - r} $$

This formula reveals the essence of convergence, where the partial sums approach a finite limit.

Analyzing the Given Data: First Term and Sum

The problem states:

- The first term of the series, $a = 7$.
- The sum of the infinite series, $S_\infty = 4$.

Our goal is to find the third term of this series, which is ar^2.

Determining the Common Ratio r

Using the sum formula for an infinite geometric series:

$$S_\infty = \frac{a}{1 - r}$$

Plugging in the known values:

$$4 = \frac{7}{1 - r}$$

Rearranged to solve for r:

$$1 - r = \frac{7}{4}$$

$$r = 1 - \frac{7}{4} = 1 - 1.75 = -0.75$$

This calculation reveals a key property:

- The common ratio $r = -0.75$.

The negative sign indicates the series alternates in sign, which is typical in many geometric series with ratios between -1 and 1.

Verifying the Conditions for Convergence

Since $|r| = 0.75 < 1$, the series converges, validating our use of the sum formula. Moreover, the negative ratio implies the terms oscillate, decreasing in magnitude over time.

Calculating the Third Term: Step-by-Step Process

With $a = 7$ and $r = -0.75$, computing the third term involves straightforward substitution.

Expression for the Third Term

The third term T_3 in a geometric series is:

$$T_3 = ar^2$$

Substituting the known values:

$$ T_3 = 7 \times (-0.75)^2 $$

Calculating $(-0.75)^2$:

$$ (-0.75)^2 = 0.5625 $$

Thus,

$$ T_3 = 7 \times 0.5625 = 3.9375 $$

Result: The third term of the series is 3.9375.

Implications and Insights of the Result

Having determined the third term, it's instructive to analyze what this value reveals about the series and its behavior.

Behavior of the Series

- Alternating pattern: Because r is negative, the series oscillates—terms alternate in sign. The first few terms are:

- $T_1 = 7$
- $T_2 = 7 \times -0.75 = -5.25$
- $T_3 = 3.9375$

- Decreasing magnitude: The absolute value of each term decreases by a factor of 0.75, indicating the series converges.

Convergence and Practical Applications

This specific series could model various real-world phenomena, such as:

- Damped oscillations in physics.

- Financial models involving diminishing payments.
- Signal processing where alternating signals decay over time.

Understanding the third term's magnitude and sign helps in predicting the series' future behavior and in designing systems that leverage such decay patterns.

Broader Mathematical Context and Related Concepts

The problem exemplifies broader principles in mathematical analysis and series theory.

Relation to Geometric Series Sum Formula

The key insight from the problem is how the sum formula connects the initial term and the ratio:

$$S_\infty = \frac{a}{1 - r}$$

This relationship highlights the importance of the ratio's magnitude and sign in series convergence.

Significance of the Ratio r

- When $|r| < 1$, the series converges.
- If $r \geq 1$, the series diverges.
- Negative ratios introduce oscillations, which are relevant in many physical systems.

Applications in Calculus and Analysis

Infinite geometric series form the basis for understanding power series, which are central in calculus for representing functions as infinite sums. They also underpin the Taylor and Maclaurin series expansions.

Conclusion: The Power of Algebraic Reasoning in Series Problems

This exploration demonstrates how a seemingly simple problem—finding the third term of an infinite geometric series—opens a window into rich mathematical concepts. By leveraging the formula for the sum of an infinite series, we deduced the common ratio and then used it to compute the specific term, illustrating the interconnectedness of algebra, series theory, and real-world modeling.

The key takeaway is that understanding the relationships between initial terms, ratios, and sums enables mathematicians and practitioners to analyze complex systems with elegance and precision. Whether in pure mathematics, physics, economics, or engineering, mastery of these foundational ideas empowers us to decode patterns, predict behaviors, and design systems that harness the power of infinite processes.

In sum, the third term of the series—3.9375—not only completes the specific problem but also exemplifies the broader significance of geometric series in mathematical analysis and applied sciences.

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