A Falling Object Satisfies The Initial Value Problem Dv/dt=9.8-(v/5), V(0)=0where V Is The Velocity In

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Understanding the motion of falling objects is fundamental in physics and engineering. When an object is dropped, its velocity changes over time due to gravitational acceleration and air resistance. In many cases, these dynamics can be modeled mathematically using differential equations. One such model is the initial value problem (IVP):

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad V(0) = 0$$

where $V(t)$ represents the velocity of the falling object at time t (in seconds). This differential equation encapsulates the balance between gravitational force pulling the object downward and the resistance (drag) proportional to velocity.

In this article, we will explore the meaning of this initial value problem, how to solve it, interpret the solution physically, and understand its implications in real-world scenarios. We will organize the discussion into comprehensive sections for clarity.

Understanding the Differential Equation of a Falling Object

The Physical Context

When an object is dropped from rest, its motion is influenced primarily by two forces:

- Gravity: Which accelerates the object downward at approximately 9.8 m/s^2 near Earth's surface.
- Air Resistance (Drag): Which opposes the motion and typically increases with velocity.

The differential equation:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

models the net acceleration considering these forces. Here:

- The term 9.8 represents the acceleration due to gravity.
- The term $v/5$ models the air resistance, assuming a linear relationship with velocity (valid for relatively low speeds).

This form suggests that as the velocity increases, air resistance grows, reducing the net acceleration until the object reaches terminal velocity.

Initial Conditions and Their Significance

The initial value $V(0) = 0$ indicates that at time $t=0$, the object starts from rest. This initial condition is essential for solving the differential equation uniquely and understanding how the velocity evolves from zero.

Mathematical Solution of the Initial Value Problem

Rewriting the Differential Equation

The differential equation:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

is a first-order linear ordinary differential equation. It can be rewritten as:

$$\frac{dv}{dt} + \frac{1}{5}v = 9.8$$

This form is suitable for the integrating factor method.

Finding the Integrating Factor

The integrating factor (IF) is:

$$\mu(t) = e^{\int P(t) dt}$$

where $P(t) = \frac{1}{5}$. So,

$$\mu(t) = e^{(1/5)t}$$

Solving the Differential Equation

Multiplying both sides by the integrating factor:

$$e^{(1/5)t} \frac{dv}{dt} + \frac{1}{5} e^{(1/5)t} v = 9.8 e^{(1/5)t}$$

The left side simplifies to:

$$\frac{d}{dt} \left(e^{(1/5)t} v \right) = 9.8 e^{(1/5)t}$$

Integrate both sides with respect to t:

$$e^{(1/5)t} v = \int 9.8 e^{(1/5)t} dt + C$$

Calculating the integral:

$$ \int 9.8 e^{(1/5)t} dt = 9.8 \times \frac{e^{(1/5)t}}{(1/5)} + C' = 9.8 \times 5 e^{(1/5)t} + C' = 49 e^{(1/5)t} + C' $$

Thus,

$$ e^{(1/5)t} v = 49 e^{(1/5)t} + C $$

Dividing through by $e^{(1/5)t}$:

$$ v(t) = 49 + C e^{-(1/5)t} $$

Applying the initial condition $V(0) = 0$:

$$ 0 = 49 + C e^{0} \Rightarrow C = -49 $$

Final solution:

$$ V(t) = 49 \left(1 - e^{-\frac{t}{5}} \right) $$

This function describes how the velocity of the falling object varies over time.

Physical Interpretation of the Solution

Velocity Behavior Over Time

The velocity function:

$$ V(t) = 49 \left(1 - e^{-\frac{t}{5}} \right) $$

has several key features:

- Initial velocity: At $t = 0$, $V(0) = 0$, matching the initial condition.
- Asymptotic behavior: As $t \to \infty$, $e^{-\frac{t}{5}} \to 0$, so:

$$ V(t) \to 49 \text{ m/s} $$

which is the terminal velocity—the maximum velocity the object approaches due to air resistance balancing gravity.

- Rate of approach: The exponential decay term indicates the velocity approaches terminal velocity exponentially fast, with a characteristic time constant $\tau = 5$ seconds.

Terminal Velocity and Its Significance

The terminal velocity V_{term} can be derived directly from the differential equation as the steady-state solution where acceleration is zero:

$$0 = 9.8 - \frac{V_{term}}{5} \Rightarrow V_{term} = 9.8 \times 5 = 49 \text{ m/s}$$

This is consistent with the limit of $V(t)$ as $t \to \infty$.

Applications and Implications of the Model

Real-World Scenarios

The model is applicable in various situations:

- Parachuting: Understanding how a skydiver's velocity approaches terminal velocity.
- Ballistics: Predicting the speed of objects falling through the air.
- Engineering Design: Designing objects with specific fall characteristics, such as safety helmets or sports equipment.

Limitations of the Model

While insightful, this model assumes:

- Linear air resistance proportional to velocity, valid for low speeds.
- Constant gravitational acceleration, ignoring variations with altitude.
- No other forces like wind or lift.

For high-speed or complex scenarios, more sophisticated models involving quadratic drag or variable gravity are necessary.

Summary and Key Takeaways

- The initial value problem $\frac{dv}{dt} = 9.8 - \frac{v}{5}$, $V(0) = 0$ models a falling object experiencing linear air resistance.

- The solution:

  $$V(t) = 49 \left(1 - e^{-\frac{t}{5}} \right)$$

  describes velocity growth over time, approaching terminal velocity of 49 m/s exponentially.

- Understanding this model helps predict the behavior of falling objects, optimize safety designs, and analyze physical phenomena involving drag and gravity.

- The model’s simplicity makes it a valuable educational tool, despite its limitations for more complex real-world scenarios.

Conclusion

The differential equation $\frac{dv}{dt} = 9.8 - \frac{v}{5}$ with initial condition $V(0) = 0$ provides a fundamental understanding of how objects accelerate under gravity while experiencing air resistance. The explicit solution reveals the exponential approach to terminal velocity, illustrating key concepts in differential equations, physics, and engineering. Recognizing the limitations and applications of this model enhances its utility in both academic and practical contexts, making it an essential topic in the study of dynamics.

Frequently Asked Questions

What does the differential equation dv/dt = 9.8 - (v/5) represent in the context of a falling object?

It models the velocity of a falling object considering gravity (9.8 m/s^2) and a damping term proportional to velocity (v/5), representing air resistance or drag effects.

What is the initial condition V(0) = 0 indicating for the falling object?

It indicates that the object starts from rest at time t = 0.

How do you solve the differential equation dv/dt = 9.8 - (v/5)?

It's a first-order linear ODE. You can solve it using integrating factors or separation of variables to find the velocity function V(t).

What is the general solution for V(t) given the initial condition V(0)=0?

The solution is V(t) = 49 (1 - e^(-t/5)), which describes the velocity of the falling object over time.

What is the significance of the exponential term in the velocity solution?

The exponential term e^(-t/5) shows how the velocity approaches a terminal velocity of 49 m/s over time due to air resistance.

How is the terminal velocity determined from the differential equation?

Terminal velocity occurs when dv/dt = 0, so setting 0 = 9.8 - (v/5) gives v = 49 m/s.

What is the physical interpretation of the parameters in the differential equation?

The parameter 9.8 represents acceleration due to gravity, while 1/5 relates to the damping effect of air resistance on the falling object.

How does the initial velocity condition affect the solution of the differential equation?

It ensures that the integration constant is set so that $V(0) = 0$, matching the initial state of the object starting from rest.

Can this model be used to describe real-world falling objects accurately?

Yes, especially for objects where air resistance is proportional to velocity; however, more complex drag models may be needed for precise predictions at high speeds.

What is the importance of understanding this differential equation in physics?

It helps in analyzing motion under gravity with resistance, illustrating concepts like terminal velocity and dynamic equilibrium in falling objects.

Additional Resources

A Falling Object Satisfies the Initial Value Problem Dv/dt=9.8-(v/5), V(0)=0 — this statement encapsulates a fundamental concept in physics and differential equations, illustrating how velocity evolves over time under the influence of gravity and air resistance. Understanding this initial value problem (IVP) is crucial for students, engineers, and physicists interested in modeling real-world phenomena like falling objects. In this comprehensive guide, we will explore the problem's formulation, solve the differential equation step-by-step, interpret the solution physically, and discuss its broader implications.

Introduction: The Significance of the Initial Value Problem in Physics

When an object is dropped from rest, its velocity doesn't instantly reach a terminal value; rather, it accelerates under gravity, with air resistance gradually influencing its acceleration. The initial value problem described by the differential equation:

$$ \frac{dv}{dt} = 9.8 - \frac{v}{5} $$

with the initial condition

$$ V(0) = 0 $$

captures this dynamic. The key components here include:

- Gravity term ($9.8\ m/s^2$): Represents the acceleration due to gravity.
- Damping term $-v/5$: Models air resistance, proportional to velocity.
- Initial velocity $V(0)=0$: The object starts from rest.

Understanding how to solve this IVP provides insights into the velocity profile of a falling object and illustrates the application of differential equations in modeling physical systems.

Breaking Down the Differential Equation

Interpreting the Equation

The differential equation:

$$ \frac{dv}{dt} = 9.8 - \frac{v}{5} $$

can be viewed as a first-order linear ODE, where:

- The forcing term is a constant (9.8), representing gravity.
- The damping term $-v/5$ accounts for air resistance proportional to the current velocity.

This form suggests that as the object accelerates, air resistance opposes the motion, eventually leading to a constant velocity (terminal velocity).

Physical Significance

- When $\frac{dv}{dt} = 0$, the velocity reaches terminal velocity v_t:

$$ 0 = 9.8 - \frac{v_t}{5} \Rightarrow v_t = 9.8 \times 5 = 49\, \text{m/s} $$

- The solution will show how the velocity approaches this terminal velocity over time.

Step-by-Step Solution of the Initial Value Problem

1. Recognize the Type of Differential Equation

The equation:

$$ \frac{dv}{dt} + \frac{1}{5}v = 9.8 $$

is a linear first-order ODE in standard form:

$$ \frac{dv}{dt} + p(t) v = q(t) $$

with

- $p(t) = \frac{1}{5}$
- $q(t) = 9.8$

2. Find the Integrating Factor (IF)

The integrating factor:

$$\mu(t) = e^{\int p(t) dt} = e^{\int \frac{1}{5} dt} = e^{t/5}$$

3. Multiply Through by the Integrating Factor

$$
e^{t/5} \frac{dv}{dt} + e^{t/5} \times \frac{1}{5} v = 9.8 e^{t/5}
$$

This simplifies to:

$$
\frac{d}{dt} \left(e^{t/5} v \right) = 9.8 e^{t/5}
$$

4. Integrate Both Sides

$$
\int \frac{d}{dt} \left(e^{t/5} v \right) dt = \int 9.8 e^{t/5} dt
$$

The left side integrates directly to:

$$
e^{t/5} v
$$

The right side requires integration:

$$
\int 9.8 e^{t/5} dt
$$

Let’s perform this integral:

$$
\int e^{t/5} dt
$$

Set:

$$
u = t/5 \Rightarrow du = dt/5 \Rightarrow dt = 5 du
$$

Thus:

$$
\int e^{u} \times 5 du = 5 \int e^{u} du = 5 e^{u} + C = 5 e^{t/5} + C
$$

Multiplying by 9.8:

$$
9.8 \times 5 e^{t/5} = 49 e^{t/5}
$$

So,

$$
e^{t/5} v = 49 e^{t/5} + C
$$

5. Solve for $v(t)$:

$$
v(t) = 49 + C e^{-t/5}
$$

6. Apply Initial Condition $V(0) = 0$:

$$
v(0) = 49 + C e^{0} = 49 + C = 0
$$

$$
C = -49
$$

Final Velocity Function:

$$
\boxed{
V(t) = 49 \left(1 - e^{-t/5}\right)
}
$$

This formula describes how the velocity of the falling object evolves over time.

Physical Interpretation of the Solution

Velocity Behavior Over Time

- At $t=0$: $V(0) = 0$, matching the initial condition.

- As $t \to \infty$: $e^{-t/5} \to 0$, so

$$
V(t) \to 49\, \text{m/s}
$$

which is the terminal velocity.

Time to Reach Near Terminal Velocity

- To estimate how quickly the object approaches terminal velocity, note that:

$$
V(t) \approx 49 \left(1 - e^{-t/5}\right)
$$

- For example, at $t=15\, \text{s}$:

$$
V(15) \approx 49 (1 - e^{-3}) \approx 49 (1 - 0.05) \approx 46.3\, \text{m/s}
$$

- The velocity reaches about 95% of terminal velocity around $t=3 \times 5 = 15\, \text{s}$, illustrating the exponential approach characteristic of such systems.

Broader Implications and Applications

Modeling Real-World Falling Objects

This differential equation provides an idealized model of a falling object's velocity considering air resistance proportional to velocity—a common assumption in physics for moderate speeds.

- Terminal velocity: The maximum speed an object reaches when gravity's pull balances air resistance.
- Design considerations: Engineers designing parachutes, vehicles, or projectiles use similar models to predict behavior.

Extensions and Variations

- Different damping models: Nonlinear air resistance (quadratic in velocity) for high-speed objects.
- Variable gravity: Adjusting for planetary environments.
- Additional forces: Including buoyancy or lift effects.

Summary and Key Takeaways

- The initial value problem:

$$
\frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad V(0) = 0
$$

models the velocity of a falling object with air resistance proportional to velocity.

- The solution:

$$
V(t) = 49 \left(1 - e^{-t/5}\right)
$$

shows the velocity starting at zero and asymptotically approaching the terminal velocity of 49 m/s.

- The exponential decay in the solution reflects how air resistance gradually balances gravity, resulting in a smooth acceleration to terminal velocity.

- This problem exemplifies the power of differential equations in modeling physical phenomena and can be extended to more complex systems.

Final Thoughts

Understanding how a falling object satisfies the initial value problem $\frac{dv}{dt} = 9.8 - \frac{v}{5}$, $V(0) = 0$, provides a foundational insight into dynamics and differential equations. It demonstrates how mathematical modeling captures real-world behavior, offering predictive power and deeper comprehension of physical laws. Whether in physics, engineering, or applied mathematics, mastering such problems forms a cornerstone for analyzing systems influenced by forces and damping effects.

Note: Always verify solutions by substituting back into the original differential equation and initial conditions to ensure consistency and correctness.

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