A Polynomial Of Degree 3,P(x) Has Leading Coefficient 1 , And Has Roots Of 7 I And 2 . Find A Possible

A Polynomial Of Degree 3, P(x), Has Leading Coefficient 1, And Has Roots Of 7i And 2. Find A Possible

When exploring polynomial functions, especially those of degree 3, understanding how roots influence the polynomial's structure is fundamental. The problem states that the polynomial P(x) is of degree 3, has a leading coefficient of 1, and roots at 7i and 2. Our goal is to find a possible form of this polynomial. This type of problem combines knowledge of complex roots, polynomial factors, and the fundamental theorem of algebra, providing a comprehensive example of polynomial construction.

In this article, we will explore step-by-step how to determine a polynomial that satisfies these conditions. We will analyze roots, conjugate roots, and how to construct the polynomial from these roots, culminating in a complete example.

Understanding the Roots and the Polynomial Structure

Roots of the Polynomial

Given roots:
- 7i (a complex root)
- 2 (a real root)

Because the polynomial has real coefficients (implied, as most polynomial problems of this type assume real coefficients unless otherwise specified), complex roots must occur in conjugate pairs. This means:

- If 7i is a root, then its conjugate, -7i, must also be a root.

Thus, the roots of the polynomial are:
- 2
- 7i
- -7i

Implication for the Polynomial

Since the polynomial is degree 3 (cubic), and we have identified three roots, the roots account for the entire polynomial.

Given the roots, the polynomial can be expressed as:

$$P(x) = (x - r_1)(x - r_2)(x - r_3)$$

where r_1, r_2, r_3 are the roots.

Constructing the Polynomial From Roots

Step 1: Write the Factors Corresponding to Roots

Based on the roots identified:

- For root 2: $(x - 2)$
- For root 7i: $(x - 7i)$
- For root -7i: $(x + 7i)$

The polynomial, before expansion, is:

$$P(x) = (x - 2)(x - 7i)(x + 7i)$$

Step 2: Simplify the Complex Conjugate Pair

The product of the conjugate pair:

$$(x - 7i)(x + 7i)$$

is a difference of squares:

$$(x)^2 - (7i)^2$$

Calculate $(7i)^2$:

$$(7i)^2 = 49i^2$$

Recall that $i^2 = -1$, so:

$$49i^2 = 49 \times (-1) = -49$$

Thus,

$$ (x - 7i)(x + 7i) = x^2 - (-49) = x^2 + 49 $$

Forming the Complete Polynomial

Now, the polynomial is:

$$ P(x) = (x - 2)(x^2 + 49) $$

Expanding this product:

$$ P(x) = x(x^2 + 49) - 2(x^2 + 49) $$

$$ P(x) = x^3 + 49x - 2x^2 - 98 $$

Rearranged in standard polynomial form:

$$ P(x) = x^3 - 2x^2 + 49x - 98 $$

This polynomial has the following properties:
- Degree 3
- Leading coefficient 1
- Roots at 2, 7i, and -7i

Verifying the Polynomial

Check Roots

- Root at 2: Plug into $P(x)$:

$$ P(2) = 2^3 - 2 \times 2^2 + 49 \times 2 - 98 = 8 - 2 \times 4 + 98 - 98 = 8 - 8 + 98 - 98 = 0 $$

- Root at 7i:

$$ P(7i) = (7i)^3 - 2(7i)^2 + 49(7i) - 98 $$

Calculate each term:

- $(7i)^3 = 7^3 i^3 = 343 i^3$

Since $i^3 = i^2 \times i = (-1) \times i = -i$:

$$343 i^3 = 343 \times (-i) = -343 i$$

- $(7i)^2 = 49 i^2 = 49 \times (-1) = -49$

- $49 \times 7i = 343 i$

Putting it all together:

$$P(7i) = -343 i - 2 \times (-49) + 343 i - 98 = -343 i + 98 + 343 i - 98 = (-343 i + 343 i) + (98 - 98) = 0 + 0 = 0$$

Similarly, for $-7i$, the calculation will also yield zero, confirming these roots.

Summary of the Polynomial Construction Process

The process to find a polynomial of degree 3 with leading coefficient 1, roots at 7i and 2, involves:

1. Recognizing the conjugate root $-7i$ due to real coefficients.
2. Expressing the polynomial as the product of factors corresponding to the roots:

$$P(x) = (x - 2)(x - 7i)(x + 7i)$$

3. Simplifying the conjugate pair as a quadratic:

$$(x - 7i)(x + 7i) = x^2 + 49$$

4. Multiplying out all factors to get the polynomial in standard form:

$$P(x) = (x - 2)(x^2 + 49) = x^3 - 2x^2 + 49x - 98$$

Additional Considerations and Variations

Other Possible Polynomials

The polynomial derived above is one possible form. Since the problem only specifies roots, and the leading coefficient, any polynomial with those roots and leading coefficient 1 can be obtained by multiplying the polynomial by a non-zero constant. However, to keep the leading coefficient as 1, the polynomial must be as above.

Changing Roots or Leading Coefficients

- If the leading coefficient was not 1, say $a \neq 0$, then the polynomial would be:

$$P(x) = a(x - 2)(x - 7i)(x + 7i)$$

- The roots would remain the same, but the polynomial's scale would change accordingly.

Implications for Polynomial Graphs

Understanding the roots' nature allows us to sketch the polynomial's graph:

- Roots at $x = 2$, $x = 7i$, and $x = -7i$. Since $7i$ and $-7i$ are complex, they do not correspond to real x-intercepts, but they influence the polynomial's behavior in the complex plane.
- The real root at $x=2$ indicates the polynomial crosses the x-axis at $x=2$.

Conclusion

Constructing a polynomial with specific roots and properties involves a systematic process rooted in algebraic principles. For the problem at hand, recognizing the conjugate roots and expressing the polynomial as a product of factors simplifies the process. The resulting polynomial:

$$P(x) = x^3 - 2x^2 + 49x - 98$$

satisfies all the conditions: degree 3, leading coefficient 1, and roots at 2, 7i, and -7i. Such problems exemplify the importance of understanding complex roots, polynomial factorization, and the fundamental theorem of algebra, all of which are essential skills in higher-level mathematics and algebraic problem-solving.

Meta-Note: This article provides a comprehensive approach to constructing a polynomial with given roots and properties, serving as a useful guide for students and enthusiasts aiming to deepen their understanding of polynomial functions and root analysis.

Frequently Asked Questions

What is the general form of a cubic polynomial with leading coefficient 1 and roots 7i and 2?

The polynomial can be written as $P(x) = (x - 7i)(x + 7i)(x - 2)$, since complex roots come in conjugate pairs, but since only 7i is given, a possible polynomial is $P(x) = (x - 7i)(x + 7i)(x - 2)$.

How do you find a real polynomial with roots 7i and 2 and leading coefficient 1?

First, write the quadratic factor for the complex conjugate roots: $(x - 7i)(x + 7i) = x^2 + 49$. Then, multiply this by $(x - 2)$ to get the cubic polynomial: $P(x) = (x^2 + 49)(x - 2)$.

What is the explicit form of the polynomial P(x) with roots 7i and 2 and leading coefficient 1?

Expanding $P(x) = (x^2 + 49)(x - 2)$: $P(x) = x^3 - 2x^2 + 49x - 98$.

Are there multiple possible cubic polynomials with the given roots and leading coefficient 1?

Yes, if the root 7i is given, its conjugate root -7i should also be included for real coefficients, resulting in a unique polynomial: $P(x) = (x - 7i)(x + 7i)(x - 2)$. Variations can occur if roots are repeated or if complex conjugates are interpreted differently, but generally, this form is standard.

How can I verify that the polynomial P(x) = x^3 - 2x^2 + 49x - 98 has roots 2 and 7i?

Substitute $x = 2$: $P(2) = 8 - 8 + 98 - 98 = 0$, so 2 is a root. Substitute $x = 7i$: $P(7i) = (7i)^3 - 2(7i)^2 + 49(7i) - 98$. Calculate step-by-step: $(7i)^3 = 343i^3 = 343i^3$, $(7i)^2 = -49$, so $P(7i) = 343i^3 - 2(-49) + 343i - 98$. Since $i^3 = -i$, $343i^3 = -343i$. So, $P(7i) = -343i + 98 + 343i - 98 = 0$, confirming 7i is a root.

Additional Resources

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Introduction

In the realm of algebra and polynomial functions, understanding the relationship between roots and coefficients is fundamental. A polynomial of degree three, or cubic polynomial, with specific roots and leading coefficients, offers a fascinating glimpse into the interconnectedness of algebraic structures. Consider a polynomial $P(x)$ that is cubic (degree 3), has a leading coefficient of 1, and roots at $7i$ and 2. The challenge lies in determining a possible form for such a polynomial, given these constraints. This task involves exploring complex roots, conjugate pairs, and polynomial factorization, providing insights into how algebraic properties guide the construction of specific polynomials.

Understanding the Foundations: Polynomial Roots and Coefficients

Before delving into the problem at hand, it's important to review the foundational principles that connect roots and coefficients in polynomial functions:

- Degree and Roots: A polynomial of degree n has exactly n roots (including complex roots and multiplicities). For a cubic polynomial, there are three roots, counted with multiplicity.

- Leading Coefficient and Monic Polynomials: When the leading coefficient (the coefficient of the highest degree term) is 1, the polynomial is called monic. This simplifies the relationships between roots and coefficients, as the polynomial can be expressed as:

$$
P(x) = x^3 + a_2 x^2 + a_1 x + a_0
$$

- Fundamental Theorem of Algebra: Ensures a polynomial of degree n has exactly n roots in the complex plane, counting multiplicities.

- Complex Roots and Conjugate Pairs: If a polynomial has real coefficients and a complex root $a + bi$, then its conjugate $a - bi$ must also be a root to ensure the coefficients remain real.

Given these principles, the problem specifies roots at $7i$ and 2, with the polynomial being monic and cubic. This indicates that:

- One root is at 2, which is real.
- Another root at $7i$, which is purely imaginary.
- The third root must then be determined, considering the conjugate root principle.

Roots of the Polynomial: Analyzing the Given and Implied Roots

The problem states that the roots include $7i$ and 2. Since the polynomial is monic and likely has real coefficients (a common assumption unless otherwise specified), the roots must satisfy certain symmetry properties:

- Root at $7i$: Since this root is purely imaginary, its conjugate $-7i$ must also be a root to maintain real coefficients.

- Root at 2: Being real, this root does not require a conjugate.

Therefore, the roots are:

$$
\boxed{
\text{Roots: } 2, \quad 7i, \quad -7i
}
$$

This set of roots ensures that the polynomial has real coefficients, and the roots align with the fundamental theorem of algebra.

Constructing the Polynomial: From Roots to Factors

Knowing the roots, the polynomial can be constructed as a product of linear factors corresponding to each root:

$$
P(x) = (x - r_1)(x - r_2)(x - r_3)
$$

Where:

- $r_1 = 2$
- $r_2 = 7i$
- $r_3 = -7i$

Thus,

$$
P(x) = (x - 2)(x - 7i)(x + 7i)

$$
$$

Note that $(x - 7i)(x + 7i)$ is a difference of squares:

$$
(x - 7i)(x + 7i) = x^2 - (7i)^2
$$

Calculating $(7i)^2$:

$$
(7i)^2 = 7^2 \times i^2 = 49 \times (-1) = -49
$$

Therefore,

$$
(x - 7i)(x + 7i) = x^2 - (-49) = x^2 + 49
$$

Putting it all together:

$$
P(x) = (x - 2)(x^2 + 49)
$$

Expanding this product:

$$
P(x) = x(x^2 + 49) - 2(x^2 + 49) = x^3 + 49x - 2x^2 - 98
$$

Rearranged into standard polynomial form:

$$
\boxed{
P(x) = x^3 - 2x^2 + 49x - 98
}
$$

This cubic polynomial has degree 3, leading coefficient 1, and roots at $2, 7i, -7i$, satisfying all the constraints.

Validation of the Polynomial: Confirming the Roots

To ensure the constructed polynomial is valid, verify that it has the intended roots:

- At $x = 2$:

$$
P(2) = 2^3 - 2 \times 2^2 + 49 \times 2 - 98 = 8 - 8 + 98 - 98 = 0
$$

- At $x = 7i$:

$$
P(7i) = (7i)^3 - 2(7i)^2 + 49(7i) - 98
$$

Calculating each term:

$$
(7i)^3 = 7^3 i^3 = 343 i^3
$$
$$
i^3 = i^2 \times i = (-1) \times i = -i
$$

So,

$$
(7i)^3 = 343 \times (-i) = -343 i
$$

Next,

$$
(7i)^2 = 49 i^2 = 49 \times (-1) = -49
$$

Consequently,

$$
P(7i) = -343 i - 2 \times (-49) + 49 \times 7i - 98
$$

Simplify each term:

$$
-343 i + 98 + 343 i - 98
$$

Adding these:

$$
(-343 i + 343 i) + (98 - 98) = 0 + 0 = 0
$$

Similarly, for $x = -7i$:

$$
P(-7i) = (-7i)^3 - 2(-7i)^2 + 49(-7i) - 98
$$

Calculations:

$$
(-7i)^3 = -7^3 i^3 = -343 \times (-i) = 343 i
$$

$$
(-7i)^2 = 49 i^2 = -49
$$

Substituting:

$$
P(-7i) = 343 i - 2 \times (-49) - 343 i - 98
$$

Simplify:

$$
343 i + 98 - 343 i - 98 = (343 i - 343 i) + (98 - 98) = 0 + 0 = 0
$$

Thus, the polynomial has the roots at 2, $7i$, and $-7i$ as required.

Broader Implications and Variations

The process outlined illustrates how roots directly influence the polynomial's structure, especially when dealing with complex roots. Several key takeaways include:

- Conjugate Roots: For polynomials with real coefficients, complex roots come in conjugate pairs, ensuring the polynomial remains real.

- Leading Coefficient: A monic polynomial simplifies the relationship between roots and coefficients, with the roots determining the polynomial uniquely up to the constant term.

- Constructive Approach: Starting from roots, one can build the polynomial systematically by forming linear factors, then expanding and simplifying.

- Multiple Roots and Multiplicities: If roots have multiplicities greater than one, the factors are raised to higher powers, affecting the polynomial's form.

In more advanced contexts, such as control theory or signal processing, understanding how roots (or zeros) influence polynomial behavior is crucial for system stability and filter design.

Final Remarks

Constructing a polynomial with specific roots and constraints is a foundational skill in algebra, with applications spanning mathematics, engineering, and physics. The example explored here demonstrates how complex roots can be incorporated into polynomial functions through conjugate pairs, ensuring real coefficients. By methodically translating roots into factors and expanding, one can derive explicit polynomial forms that satisfy given conditions.

This approach not only aids in solving particular problems but also deepens the understanding of the fundamental relationship between roots and coefficients, a cornerstone of algebraic theory. Whether tackling academic exercises or practical engineering problems, mastering these techniques equips learners and professionals with a powerful toolset for polynomial analysis.

Summary

- The polynomial \(P(x) \) is monic (leading coefficient 1) and degree 3.
- Roots are at \(2 \), \(7i \), and \(-7i \).
- The polynomial can be expressed as \(P(x) = (x - 2)(x^2 + 49) \).
- Expanding yields \(P(x) = x

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