A Set Of 5 Vectors In R4 Is Given. Are They Linearly Dependent? Do They Span R4? Do They Form A Basis?

A Set Of 5 Vectors In R4 Is Given. Are They Linearly Dependent? Do They Span R4? Do They Form A Basis?

Understanding the properties of vectors in $\mathbb{R}^4$ is fundamental in linear algebra. When dealing with a set of vectors, especially in higher-dimensional spaces like $\mathbb{R}^4$, it is crucial to analyze their linear dependence or independence, whether they span the entire space, and if they constitute a basis. In this article, we examine these concepts thoroughly, considering a specific set of five vectors in $\mathbb{R}^4$, and explore the implications of their relationships.

Fundamental Concepts in Vector Spaces

Before delving into the specific questions, it is essential to clarify some core concepts that underpin the analysis.

Linear Independence and Dependence

- Linear Independence: A set of vectors $\{v_1, v_2, ..., v_k\}$ in $\mathbb{R}^n$ is said to be linearly independent if the only solution to the equation

$$
c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0
$$

is when all coefficients $c_i = 0$.

- Linear Dependence: Conversely, the set is linearly dependent if there exists a non-trivial combination (not all $c_i = 0$) such that the linear combination equals zero.

- Implication in $\mathbb{R}^4$: Since $\mathbb{R}^4$ is 4-dimensional, any set of more than 4 vectors is necessarily linearly dependent, as per the dimension theorem.

Spanning a Vector Space

- A set of vectors $\{v_1, v_2, ..., v_k\}$ spans $\mathbb{R}^4$ if every vector in

\(\mathbb{R}^4\) can be expressed as a linear combination of the vectors.

- To determine if a set spans \(\mathbb{R}^4\), we check whether the vectors' linear combinations can generate any arbitrary vector in \(\mathbb{R}^4\).

Basis of a Vector Space

- A basis is a set of vectors that is both linearly independent and spanning the space.

- In \(\mathbb{R}^4\), a basis consists of exactly 4 vectors.

- Whether a set of vectors forms a basis depends on these two properties.

Analyzing a Set of 5 Vectors in \(\mathbb{R}^4\)

Suppose the set of vectors is:

\[
V = \{v_1, v_2, v_3, v_4, v_5\}
\]

where each \(v_i \in \mathbb{R}^4\). The key questions are:

- Are these vectors linearly dependent?
- Do they span \(\mathbb{R}^4\)?
- Do they form a basis of \(\mathbb{R}^4\)?

Are the 5 Vectors Linearly Dependent?

Theoretical Background

- Since \(\mathbb{R}^4\) is 4-dimensional, any set of more than 4 vectors must be linearly dependent. This is a consequence of the Dimension Theorem.

- Dimension Theorem: Any set of vectors in an \(n\)-dimensional vector space containing more than \(n\) vectors is necessarily linearly dependent.

Application to the Set of 5 Vectors

- Because there are 5 vectors in $\mathbb{R}^4$, they cannot all be linearly independent.

- Therefore, the set V is necessarily linearly dependent.

Implication

- The linear dependence means at least one vector in the set can be expressed as a linear combination of others.

- This reduces the effective "independent" vectors to at most 4.

Do These Vectors Span $\mathbb{R}^4$?

Understanding Spanning Sets

- To determine whether the set $V = \{v_1, v_2, v_3, v_4, v_5\}$ spans $\mathbb{R}^4$, we need to see if their linear combinations can generate any vector in $\mathbb{R}^4$.

Key Points

- Number of vectors required to span $\mathbb{R}^4$: Exactly 4 linearly independent vectors.

- Since the set has 5 vectors and is dependent, the maximum possible rank (the number of linearly independent vectors in the set) is at most 4.

- If the 4 largest linearly independent vectors in the set span $\mathbb{R}^4$, then the entire set, including the fifth, also spans $\mathbb{R}^4$.

Practical Approach to Check Spanning

1. Form a matrix with the vectors as columns:

\[
M = \begin{bmatrix}

| & | & | & | & | \\
v_1 & v_2 & v_3 & v_4 & v_5 \\
| & | & | & | & |
\end{bmatrix}
\]

2. Perform row reduction to determine the rank.

3. Interpretation:

- If the rank is 4 (full rank for \(\mathbb{R}^4\)), then the vectors span \(\mathbb{R}^4\).

- If the rank is less than 4, they do not span \(\mathbb{R}^4\).

Conclusion

- Given that the set has more than 4 vectors, and the maximum independent subset is of size 4, the set will span \(\mathbb{R}^4\) only if these 4 independent vectors span the space. The fifth vector, being dependent, won't prevent the set from spanning, but it may be redundant.

Do the Vectors Form a Basis of \(\mathbb{R}^4\)?

Criteria for a Basis

- A set forms a basis if and only if:

1. The vectors are linearly independent.
2. The vectors span \(\mathbb{R}^4\).

Implication for the Set of 5 Vectors

- Since the set contains 5 vectors, which is more than the dimension (4), the set cannot be linearly independent.

- Therefore, it cannot form a basis for \(\mathbb{R}^4\).

Summary

- No, the set of 5 vectors in $\mathbb{R}^4$ does not form a basis.

- To form a basis, one would need exactly 4 linearly independent vectors that span $\mathbb{R}^4$.

Summary of Key Points

1. **Linear Dependence:** Any set of more than 4 vectors in $\mathbb{R}^4$ is necessarily linearly dependent.

2. **Spanning $\mathbb{R}^4$:** The set spans $\mathbb{R}^4$ if its largest linearly independent subset contains 4 vectors that span the space. The presence of a dependent vector doesn't prevent spanning, provided the independent subset spans $\mathbb{R}^4$.

3. **Forming a Basis:** The set cannot be a basis because it is not linearly independent. A basis requires exactly 4 independent vectors.

Practical Example and Computation

Suppose you are given specific vectors:

$$
v_1 = (1, 0, 0, 0), \quad v_2 = (0, 1, 0, 0), \quad v_3 = (0, 0, 1, 0), \quad v_4 = (0, 0, 0, 1), \quad v_5 = (1, 1, 1, 1)
$$

- The first four vectors are the standard basis vectors and form a basis for $\mathbb{R}^4$.

- The fifth vector is a linear combination:

$$
v_5 = v_1 + v_2 + v_3 + v_4
$$

- Since v_5 can be expressed as a linear combination of the first four, the entire set is linearly dependent.

- The set spans \(\mathbb

Frequently Asked Questions

Given a set of 5 vectors in R4, are they necessarily linearly dependent?

Yes. Since R4 is a 4-dimensional space, any set of more than 4 vectors in R4 must be linearly dependent due to the dimension theorem.

Can 5 vectors in R4 span the entire space?

Yes, if these 5 vectors are linearly independent and include a basis for R4, then they span R4. However, if they are dependent, they may or may not span R4.

Do 5 vectors in R4 form a basis for R4?

Only if they are linearly independent and span R4. Since there are more than 4 vectors, they cannot be linearly independent; thus, they cannot form a basis.

How can we determine if the given 5 vectors in R4 are linearly dependent?

By checking if the vectors satisfy a non-trivial linear combination that equals the zero vector, typically via Gaussian elimination on their matrix representation.

If the 5 vectors are linearly dependent, what does that imply about their span in R4?

It implies that their span is a subspace of R4 with dimension less than 4, so they do not span the entire space.

What is the significance of having more than 4 vectors in R4 in terms of basis and span?

Having more than 4 vectors guarantees linear dependence, so they cannot form a basis, but they may still span a subspace of R4 depending on their linear relations.

Additional Resources

Linear Dependence and Basis in R4: An In-Depth Analysis of a Set of Five Vectors

Understanding the structure and properties of vectors in \(\mathbb{R}^4\) is a fundamental aspect of linear algebra. When given a set of vectors, one of the core questions is whether they are linearly dependent or independent, whether they span the space, and whether they form a basis. In this comprehensive review, we will explore these concepts thoroughly, focusing on a set of five vectors in \(\mathbb{R}^4\).

Introduction to the Problem

Suppose you are presented with five vectors in four-dimensional space:

$$
\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5 \in \mathbb{R}^4
$$

The core questions to address are:

- Are these vectors linearly dependent or independent?
- Do they span $\mathbb{R}^4$?
- Do they form a basis for $\mathbb{R}^4$?

Understanding the answers to these questions hinges on the fundamental properties of vectors, linear combinations, and the concepts of span, independence, and basis.

Fundamental Concepts

Linear Dependence and Independence

- Linear Dependence: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is linearly dependent if there exist scalars $c_1, c_2, ..., c_k$, not all zero, such that:

$$
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k = \mathbf{0}
$$

- Linear Independence: Conversely, if the only solution to the above equation is when all $c_i = 0$, then the vectors are linearly independent.

Key Point: In $\mathbb{R}^4$, any set of more than 4 vectors must be linearly dependent because the maximum number of linearly independent vectors in $\mathbb{R}^4$ is 4.

Span of Vectors

- The span of a set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$, denoted $\text{span}\{\mathbf{v}_1, ..., \mathbf{v}_k\}$, is the set of all linear combinations:

$$
\text{span}\{\mathbf{v}_1, ..., \mathbf{v}_k\} = \left\{ c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k \mid c_i \in \mathbb{R} \right\}
$$

- To span $\mathbb{R}^4$, the set must be capable of generating any vector in $\mathbb{R}^4$.

Basis of $\mathbb{R}^4$

- A basis for $\mathbb{R}^4$ is a set of 4 vectors that are linearly independent and span $\mathbb{R}^4$.

Important: Since the basis must have exactly 4 vectors, any set with more than four vectors (like five vectors) cannot be a basis unless some vectors are redundant, i.e., linearly dependent.

Analyzing a Set of Five Vectors in $\mathbb{R}^4$

Given the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, the analysis involves several steps:

Step 1: Constructing the Matrix

Represent the vectors as columns in a matrix:

$$
A = \begin{bmatrix}
| & | & | & | & | \\
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\
| & | & | & | & |
\end{bmatrix}
$$

This is a 4×5 matrix, with each column being one of the vectors.

Step 2: Determining Linear Dependence

- Since the matrix has more columns (vectors) than rows (dimension of space), it is impossible for all five vectors to be linearly independent in $\mathbb{R}^4$.

- Conclusion: The set of five vectors must be linearly dependent. This is a consequence of the Dimension Theorem: in an n-dimensional vector space, any set of more than n

vectors is linearly dependent.

Therefore, the answer to the first question is YES, the five vectors are linearly dependent.

Step 3: Do They Span $\mathbb{R}^4$?

- To span $\mathbb{R}^4$, the set must contain at least 4 linearly independent vectors (since the dimension of $\mathbb{R}^4$ is 4).

- The five vectors include at least four vectors that may or may not be independent.

- Key Point: Because the set is larger than 4, and some vectors are dependent, it is still possible that some subset of four vectors spans $\mathbb{R}^4$.

- Method:
- Perform row operations or Gaussian elimination on the matrix A to find its rank.
- The rank tells us the size of the largest set of linearly independent vectors within the set.

- If the rank of A is 4, then the vectors do span $\mathbb{R}^4$, regardless of the fifth vector.

- If the rank is less than 4, then the vectors do not span $\mathbb{R}^4$.

Summary:
- The set spans $\mathbb{R}^4$ if and only if the rank of A is 4.

Step 4: Do They Form a Basis?

- For a set to be a basis:
1. It must be linearly independent.
2. It must span the entire space $\mathbb{R}^4$.

- Since the set has more than 4 vectors in $\mathbb{R}^4$, it cannot be linearly independent.

- Therefore, the set of five vectors cannot form a basis, because a basis must have exactly 4 vectors in $\mathbb{R}^4$.

Practical Procedure for Analysis

To analyze any specific set of five vectors, follow these steps:

1. Write the vectors as columns of a 4×5 matrix A.

2. Perform Gaussian elimination to row reduce A to its echelon form.

3. Determine the rank:

- If the rank is 4:
- The vectors span $\mathbb{R}^4$.
- The vectors are not linearly independent (since there are 5 vectors), but some subset of four vectors is independent and forms a basis.
- If the rank is less than 4:
- The vectors do not span $\mathbb{R}^4$.

4. Identify a basis:
- Find a subset of 4 vectors corresponding to the linearly independent columns identified during the reduction process.

Illustrative Example

Suppose we are given the vectors:

$$
\begin{aligned}
\mathbf{v}_1 &= (1, 0, 0, 0) \\
\mathbf{v}_2 &= (0, 1, 0, 0) \\
\mathbf{v}_3 &= (0, 0, 1, 0) \\
\mathbf{v}_4 &= (0, 0, 0, 1) \\
\mathbf{v}_5 &= (1, 1, 1, 1)
\end{aligned}
$$

Construct matrix A:

$$
A =
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
$$

Analysis:

- The first four vectors form the identity matrix columns, which are clearly linearly independent.

- The fifth vector is the sum of the first four vectors.

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