A Signal X(t) = = E^-2tu(t) Passes Through A System Whose Frequency Response Is: H(W) = {1 || < |B|

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Understanding how signals behave when passing through systems is fundamental in signal processing. When a particular signal encounters a system characterized by a specific frequency response, the output can vary significantly based on the properties of both the input signal and the system itself. In this article, we explore the behavior of the signal \(X(t) = e^{-2t} u(t)\) as it passes through a system with a frequency response described by \(H(\omega) = \{1 \text{ for } |\omega| < |B|; 0 \text{ otherwise}\}\). By examining this scenario, we can gain insights into concepts such as Fourier transforms, frequency filtering, and system analysis.

Understanding the Input Signal: \(X(t) = e^{-2t} u(t)\)

Definition and Characteristics

The input signal \(X(t) = e^{-2t} u(t)\) is a decaying exponential multiplied by the unit step function \(u(t)\). This function is zero for \(t < 0\), and for \(t \geq 0\), it exhibits exponential decay with a rate of 2:

\[
X(t) =
\begin{cases}
0, & t < 0 \\
e^{-2t}, & t \geq 0
\end{cases}
\]

This type of signal is common in systems where a process starts at \(t=0\) and decays over time, such as in RC circuits or natural decay phenomena.

Frequency Domain Representation

To understand how the system modifies the signal, we need to analyze its frequency content via the Fourier transform:

\[
X(\omega) = \mathcal{F}\{X(t)\} = \int_{-\infty}^{\infty} X(t) e^{-j \omega t} dt
\]

Since $X(t) = 0$ for $t < 0$:

$$
X(\omega) = \int_{0}^{\infty} e^{-2t} e^{-j \omega t} dt = \int_{0}^{\infty} e^{-(2 + j \omega)t} dt
$$

Evaluating the integral:

$$
X(\omega) = \frac{1}{2 + j \omega}
$$

This complex frequency response indicates that the signal contains a continuous spectrum with magnitude:

$$
|X(\omega)| = \frac{1}{\sqrt{4 + \omega^2}}
$$

and phase:

$$
\angle X(\omega) = - \arctan \left(\frac{\omega}{2}\right)
$$

System's Frequency Response: $H(\omega) = \{1 \text{ for } |\omega| < |B|; 0 \text{ otherwise}\}$

Ideal Low-Pass Filter Characteristics

The given system's frequency response $H(\omega)$ is that of an ideal low-pass filter with cutoff frequency $|B|$:

- For $|\omega| < |B|$, $H(\omega) = 1$
- For $|\omega| \geq |B|$, $H(\omega) = 0$

This filter allows frequencies below $|B|$ to pass unchanged, while completely attenuating higher frequencies.

Implications of the Filter

This ideal filter is a theoretical model used to illustrate filtering effects:

- It completely preserves the low-frequency components.
- It completely removes the high-frequency components.

- Its frequency response is a rectangular function, leading to certain artifacts like ringing in the time domain (Gibbs phenomenon).

Analyzing the Output Signal

Frequency Domain Approach

The output in the frequency domain is obtained by multiplying the input's Fourier transform with the system's frequency response:

$$
Y(\omega) = X(\omega) \times H(\omega)
$$

Given that $H(\omega)$ is 1 for $|\omega| < |B|$ and 0 elsewhere:

$$
Y(\omega) =
\begin{cases}
\frac{1}{2 + j \omega}, & |\omega| < |B| \\
0, & |\omega| \geq |B|
\end{cases}
$$

This means the output spectrum is a band-limited version of the input spectrum.

Time Domain Representation of the Output

To find the time domain response $y(t)$, we perform the inverse Fourier transform:

$$
y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j \omega t} d\omega
$$

Since $Y(\omega)$ is non-zero only in $[-|B|, |B|]$:

$$
y(t) = \frac{1}{2\pi} \int_{- |B|}^{|B|} \frac{1}{2 + j \omega} e^{j \omega t} d\omega
$$

This integral can be evaluated using contour integration or numerical methods, but qualitatively, the output is a band-limited version of the original exponential decay.

Effects of Filtering on the Signal

Attenuation and Preservation of Frequencies

The ideal low-pass filter preserves low-frequency components:

- If $|B|$ is greater than the dominant frequency of $X(t)$, most of the signal's energy passes through, resulting in a filtered signal similar to the original.
- If $|B|$ is smaller, significant parts of the signal are cut off, leading to distortion.

Time Domain Artifacts

Applying an ideal rectangular filter in frequency domain is equivalent to convolving the original signal with a sinc function in the time domain:

$$
\text{Sinc}(t) = \frac{\sin (|B| t)}{\pi t}
$$

This causes ringing artifacts, which are oscillations near sharp transitions in the time domain. The narrower the cutoff frequency $|B|$, the wider the sinc function in time, leading to more pronounced ringing.

Practical Applications and Limitations

Applications of Such Filtering

Understanding this filtering process is vital in various fields:

- Communications: Removing high-frequency noise.
- Signal analysis: Isolating relevant frequency bands.
- Audio processing: Bass boosting or treble reduction.
- Control systems: Filtering sensor signals.

Limitations of the Ideal Filter Model

While ideal filters are conceptually straightforward, they have limitations:

- Non-causal: Cannot be physically realized because they require future knowledge.
- Infinite impulse response: The sinc function extends infinitely in time.
- Gibbs phenomenon: Oscillations caused by sharp cutoff boundaries.

Conclusion: Summary of Key Concepts

In summary, passing the exponential decay signal $X(t) = e^{-2t} u(t)$ through an ideal low-pass filter with cutoff $|B|$ results in a band-limited version of the original signal. The key points include:

- The Fourier transform of the input signal is $X(\omega) = 1/(2 + j \omega)$.
- The system's frequency response acts as a filter, passing frequencies below $|B|$ and removing higher frequencies.
- The output spectrum is truncated, leading to a time domain response that can be derived via inverse Fourier transform.
- The filtering introduces artifacts such as ringing, which are inherent to ideal filters with sharp cutoffs.
- Practical considerations involve trade-offs between filter sharpness and real-world limitations.

Understanding these principles allows engineers and scientists to design appropriate filtering strategies for signal analysis, communication systems, and many other applications in modern technology.

Frequently Asked Questions

What is the time-domain expression of the input signal X(t) = E^{-2t} u(t)?

The input signal is an exponentially decaying function multiplied by the unit step function, starting at t=0: $X(t) = e^{-2t}$ for $t \geq 0$, and 0 otherwise.

What is the frequency response H(ω) of the system and what does the condition |ω| < B imply?

The frequency response is given as $H(\omega) = 1$ for $|\omega| < B$, and zero otherwise, indicating a ideal low-pass filter with cutoff frequency B.

How do you determine the output Y(t) of the system for the given input and frequency response?

Find the Fourier Transform of $X(t)$, multiply it by $H(\omega)$, and then perform the inverse Fourier Transform to obtain $Y(t)$.

What is the Fourier Transform of the input signal X(t) = e^{-2t} u(t)?

The Fourier Transform is $X(\omega) = 1 / (2 + j\omega)$.

How does the ideal low-pass filter H(ω) affect the spectrum of X(ω)?

It retains the spectrum for $|\omega| < B$ and zeroes out frequencies outside this range, effectively filtering the higher frequency components.

What is the resulting output Y(t) after passing X(t) through this system?

Y(t) is obtained by inverse Fourier transforming the product of X(ω) and H(ω); it results in a time-domain signal corresponding to the filtered spectrum.

How does the cutoff frequency B influence the output signal in terms of signal distortion?

A smaller B results in more signal attenuation and distortion of the higher frequency components, affecting the shape of the output waveform.

Can the output Y(t) be expressed in a closed-form formula? If so, how?

Yes, by performing the inverse Fourier transform of the product X(ω)H(ω), which involves integrating over the passband and using known transform pairs, resulting in a piecewise or integral expression depending on B.

Additional Resources

A Signal X(t) = e^{-2t} u(t) Passes Through A System Whose Frequency Response Is: H(ω) = {1, |ω| < |B|}

Introduction

In the realm of signal processing, understanding how signals behave as they traverse through systems is fundamental. When a signal encounters a system characterized by a particular frequency response, its amplitude and phase can be altered, leading to various effects that are crucial in applications ranging from communications to control systems. Today, we explore a specific scenario: a signal defined as X(t) = e^{-2t} u(t) passing through a system with a frequency response H(ω) = {1, |ω| < |B|}. This situation exemplifies how an exponential decay signal interacts with a band-limited system, revealing key insights into filtering, spectral analysis, and signal behavior.

Understanding the Input Signal: X(t) = e^{-2t} u(t)

Before delving into the system's influence, it is essential to analyze the input signal's nature.

What is $X(t)$?

- Expression: $X(t) = e^{-2t} u(t)$
- Components:
- Exponential decay: e^{-2t}
- Unit step function: $u(t)$, which equals 0 for $t < 0$ and 1 for $t \geq 0$

This signal is a causal exponential decay starting at $t = 0$. It is a common model in physics and engineering to represent processes like charging/discharging in RC circuits, damping in mechanical systems, or decay of a radioactive substance.

Key Properties

- Causality: Due to the $u(t)$, $X(t)$ is zero for $t < 0$.
- Time-domain behavior: The signal diminishes exponentially after $t = 0$.
- Energy and Power:
- The signal is finite energy because it is exponentially decaying.
- It is not periodic; it is a transient signal.

Fourier Transform of $X(t)$

The Fourier transform provides the spectral content of the signal:

$$
X(\omega) = \int_{-\infty}^\infty X(t) e^{-j \omega t} dt
$$

Given $X(t) = e^{-2t} u(t)$:

$$
X(\omega) = \int_{0}^\infty e^{-2t} e^{-j \omega t} dt = \int_{0}^\infty e^{-(2 + j \omega)t} dt
$$

Calculating the integral:

$$
X(\omega) = \frac{1}{2 + j \omega}
$$

This spectral representation indicates a Lorentzian-shaped spectrum centered at zero frequency, with a decay rate dictated by the real part.

The System's Frequency Response: $H(\omega) = \{1, |\omega| < |B|\}$

The system under consideration is characterized by its frequency response $H(\omega)$:

\[
H(\omega) =
\begin{cases}

1, & |\omega| < |B| \\
0, & |\omega| \geq |B|
\end{cases}
\]

This describes an ideal band-pass filter that allows frequencies within a specific band, centered at zero, to pass unchanged, while completely attenuating frequencies outside this band.

Key Characteristics

- Ideal Filter: No transition band; the cutoff is sharp.
- Passband: Frequencies with magnitude less than |B|.
- Stopband: Frequencies with magnitude greater than or equal to |B| are blocked.
- Implication: The system acts as a band-limiting filter, shaping the spectral content of the input signal.

Signal Transformation Through the System

The core of the analysis involves how the spectral content of X(t) is modified by the system's frequency response. This process is understood through the output spectrum:

\[
Y(\omega) = X(\omega) \times H(\omega)
\]

and subsequently, the output signal y(t) is obtained via the inverse Fourier transform of Y(ω).

Spectral Analysis: Computing the Output Spectrum

Given the spectral content of the input and the system's response:

\[
X(\omega) = \frac{1}{2 + j \omega}
\]

\[
H(\omega) =
\begin{cases}
1, & |\omega| < |B| \\
0, & |\omega| \geq |B|
\end{cases}
\]

The output spectrum:

\[
Y(\omega) = \frac{1}{2 + j \omega} \times H(\omega)

\]

which simplifies to:

$$
Y(\omega) =
\begin{cases}
\frac{1}{2 + j \omega}, & |\omega| < |B| \\
0, & |\omega| \geq |B|
\end{cases}
$$

This means the spectral content outside the passband is eliminated, while the content within the passband remains unchanged.

Reconstructing the Time-Domain Output Signal

The inverse Fourier transform yields the output signal:

$$
y(t) = \frac{1}{2\pi} \int_{-\infty}^\infty Y(\omega) e^{j \omega t} d\omega
$$

Given the band-limited spectral content, the integral reduces to:

$$
y(t) = \frac{1}{2\pi} \int_{- |B|}^{|B|} \frac{1}{2 + j \omega} e^{j \omega t} d\omega
$$

This integral encapsulates how the original exponential decay is modified by the filter.

Practical Implications and Insights

1. Signal Distortion and Filtering Effect

The original exponential decay contains a broad spectrum of frequencies, including those outside the passband. When passed through the band-limited system:

- High-frequency components ($|\omega| \geq |B|$) are suppressed.
- Low-frequency components ($|\omega| < |B|$) pass through unaffected.

This spectral truncation causes the output to be a filtered version of the original signal, often resulting in:

- Smoothing or smoothing artifacts.
- Loss of sharp features associated with high-frequency content.
- Potential for ringing effects in time domain due to abrupt spectral cutoff.

2. Time-Domain Effects

The inverse Fourier transform indicates that the output signal y(t) is a convolution of the original signal with the inverse Fourier transform of the filter, which is a sinc function:

$$
h(t) = \frac{1}{2\pi} \int_{- |B|}^{|B|} e^{j \omega t} d\omega = \frac{\sin(|B| t)}{\pi t}
$$

Thus, the output:

$$
y(t) = (X(t) h(t))
$$

which involves convolving the exponential decay with a sinc function, resulting in a band-limited decay that exhibits oscillations and slower decay compared to the original.

3. Applications in Signal Processing

This analysis has direct relevance to real-world applications:

- Filtering in communication systems: Removing undesired high-frequency noise.
- Data compression: Limiting bandwidth to reduce data size.
- Audio and image processing: Band-limiting to prevent aliasing.
- Control systems: Ensuring signals stay within operational bandwidths.

Analytical and Numerical Approaches

While the integral expression for y(t) can be complex, several methods are employed to evaluate it:

- Analytical solutions: Using complex analysis or residue calculus for specific B values.
- Numerical integration: Employing computational tools to approximate y(t) for given parameters.
- Simulation: Using software like MATLAB or Python's SciPy to simulate the time response.

Practical Considerations

1. Ideal Filters Are Theoretically Perfect

In reality, perfect band-limiting filters with sharp cutoffs are impossible due to physical limitations. Real filters have transition bands, and their responses are smoother, affecting the spectral content differently.

2. Effect of Filter Bandwidth |B|

- Narrow bandwidth (small |B|): Significant spectral truncation, leading to more pronounced time-domain oscillations.

- Wide bandwidth (large |B|): Less distortion, closer to the original exponential decay.

3. Signal-to-Noise Ratio (SNR)

Filtering can improve SNR by removing unwanted high-frequency noise but may also distort the desired signal if the bandwidth is too limited.

Summary and Conclusions

The interaction between a causal exponential decay signal and a band-limited system exemplifies core principles of signal processing. The key insights include:

- The spectral content of $X(t) = e^{-2t} u(t)$ is concentrated around low frequencies but extends infinitely.
- Passing through an ideal band-pass filter with frequency response $H(\omega)$ truncates the spectrum, resulting in a modified time-domain signal.
- The output is a convolution of the original exponential with a sinc function, leading to a smoothed, oscillatory decay.
- Practical applications depend on balancing bandwidth limitations with signal fidelity.

This example underscores the importance of understanding spectral properties, filter design, and their combined influence on real-world signals. Whether in telecommunications, audio engineering, or control systems, the principles demonstrated here form the foundation for designing systems that optimize signal integrity while mitigating noise and distortion.

In essence, the journey of the signal $X(t) = e^{-2t} u(t)$ through the band

A Signal X T E 2tu T Passes Through A System Whose Frequency Response Is H W 1 Lt B

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