A Tank Formed By Rotating Y = 4x 2 , 0 X 1 About The Y-axis Is Full Of Water. The Density Of The Water

A Tank Formed By Rotating Y = 4x^2 , 0 ≤ x ≤ 1 About The Y-axis Is Full Of Water. The Density Of The Water

Understanding the geometric and physical properties of tanks derived from mathematical curves is a fascinating intersection of calculus, physics, and engineering. In this article, we explore a specific tank formed by revolving the parabola $y = 4x^2$ around the y-axis, which results in a three-dimensional solid that is entirely filled with water. We will analyze the shape, volume, surface area, and the implications of water density within this tank, providing comprehensive insights suitable for students, engineers, and enthusiasts alike.

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1. Introduction to the Geometric Formation

Understanding the Curve $y = 4x^2$

- The parabola $y = 4x^2$ opens upward, symmetric about the y-axis.
- The domain is restricted to $0 \leq x \leq 1$, which means the parabola segment from $x=0$ to $x=1$.
- At $x=0$, $y=0$; at $x=1$, $y=4$.

Revolution About the Y-axis

- When this parabola is revolved around the y-axis, it creates a solid of revolution.
- The resulting shape resembles a bowl or tank, wider at the top and narrower at the bottom.
- This process involves generating a 3D surface by rotating the curve around the y-axis.

Physical Relevance and Applications

- Such mathematical modeling helps in designing tanks, containers, and other engineering structures.
- Understanding the volume and density helps in calculating the amount of water it can hold and the weight exerted on its walls.

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2. Mathematical Description of the Solid

Setting up the Solid of Revolution

- The parabola segment is defined by $y = 4x^2$, with x in $[0,1]$.
- To generate the solid, we revolve this curve around the y-axis.

Expressing x as a function of y

- From $y = 4x^2$, we have $x = □ (y/4)$.
- Since $x \geq 0$, $x = □ (y/4)$.

Limits of Integration

- y varies from the minimum $y=0$ (at $x=0$) to $y=4$ (at $x=1$).
- Therefore, $y □ [0, 4]$.

Cross-Sectional Radius

- At a height y, the radius $r(y)$ of the cross-section is $x = □ (y/4)$.
- The radius is thus $r(y) = □ (y/4)$.

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3. Calculating the Volume of the Water in the Tank

Using the Method of Cylindrical Shells or Disks

- Since the solid is generated by revolving around the y-axis, the disk method is most straightforward.

Volume Integral

- Volume V is given by:

$$
V = \pi \int_{0}^{4} [r(y)]^2 dy
$$

- Substituting $r(y)$:

$$
V = \pi \int_{0}^{4} \left(\sqrt{\frac{y}{4}} \right)^2 dy = \pi \int_{0}^{4} \frac{y}{4} dy
$$

Calculating the Integral

- Simplify the integrand:

$$
V = \frac{\pi}{4} \int_{0}^{4} y \, dy
$$

- Integrate:

$$
V = \frac{\pi}{4} \left[\frac{y^2}{2} \right]_0^4 = \frac{\pi}{4} \times \frac{(4)^2}{2} = \frac{\pi}{4} \times \frac{16}{2} = \frac{\pi}{4} \times 8 = 2\pi
$$

Result:
- The volume of water in the tank is 2π cubic units.

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4. Surface Area of the Tank

Calculating the Surface Area of the Curved Surface

- The surface area (excluding the base) of the solid generated by revolution is given by:

$$
A = 2\pi \int_{a}^{b} r(y) \sqrt{1 + \left(\frac{dx}{dy} \right)^2 } dy
$$

- Since x = ⍰ (y/4), then:

$$
\frac{dx}{dy} = \frac{1}{2} \times \frac{1}{\sqrt{y/4}} \times \frac{1}{4} = \frac{1}{2} \times \frac{1}{\sqrt{y/4}} \times \frac{1}{4}
$$

Wait, that calculation needs correction.

Deriving dx/dy

- x = ⍰ (y/4) = (y/4)^{1/2}
- Differentiate:

$$
\frac{dx}{dy} = \frac{1}{2} (y/4)^{-1/2} \times \frac{1}{4} = \frac{1}{2} \times \frac{1}{\sqrt{y/4}} \times \frac{1}{4}
$$

- Simplify:

$$
\frac{dx}{dy} = \frac{1}{8} \times \frac{1}{\sqrt{y/4}} = \frac{1}{8} \times \frac{1}{\frac{\sqrt{y}}{2}} = \frac{1}{8} \times \frac{2}{\sqrt{y}} = \frac{1}{4 \sqrt{y}}
$$

Surface Area Integral

- Now, the surface area:

$$
A = 2\pi \int_{0}^{4} r(y) \sqrt{1 + \left(\frac{dx}{dy} \right)^2 } dy
$$

- Recall r(y) = ⍰ (y/4) and dx/dy = 1/(4⍰ y).

- Compute $\left(\frac{dx}{dy} \right)^2$:

$$
\left(\frac{1}{4 \sqrt{y}} \right)^2 = \frac{1}{16 y}
$$

- Therefore:

$$
\sqrt{1 + \frac{1}{16 y}} = \sqrt{\frac{16 y + 1}{16 y}} = \frac{\sqrt{16 y + 1}}{4 \sqrt{y}}
$$

- The integrand becomes:

$$
r(y) \times \sqrt{1 + \left(\frac{dx}{dy} \right)^2 } = \sqrt{\frac{y}{4}} \times \frac{\sqrt{16 y + 1}}{4 \sqrt{y}} = \frac{\sqrt{\frac{y}{4}}}{1} \times \frac{\sqrt{16 y + 1}}{4 \sqrt{y}}
$$

- Simplify numerator:

$$
\sqrt{\frac{y}{4}} = \frac{\sqrt{y}}{2}
$$

- Substituting back:

$$
\frac{\sqrt{y}}{2} \times \frac{\sqrt{16 y + 1}}{4 \sqrt{y}} = \frac{\sqrt{16 y + 1}}{8}
$$

- The surface area integral simplifies to:

$$
A = 2\pi \int_{0}^{4} \frac{\sqrt{16 y + 1}}{8} dy = \frac{\pi}{4} \int_{0}^{4} \sqrt{16 y + 1} \, dy
$$

Calculating the Integral

- Let's compute:

$$
I = \int_{0}^{4} \sqrt{16 y + 1} \, dy
$$

- Use substitution:

$$
u = 16 y + 1 \Rightarrow du = 16 dy \Rightarrow dy = \frac{du}{16}
$$

- When $y=0$, $u=1$; when $y=4$, $u=16\times4 + 1=65$.

- The integral becomes:

$$

$$
I = \int_{u=1}^{65} \sqrt{u} \times \frac{du}{16} = \frac{1}{16} \int_{1}^{65} u^{1/2} du
$$

- Integrate:

$$
\frac{1}{16} \times \frac{2}{3} u^{3/2} \Big|_{1}^{65} = \frac{1}{16} \times \frac{2}{3} (65^{3/2} - 1^{3/2}) = \frac{1}{24} (65^{3/2} - 1)
$$

- Simplify $65^{3/2}$:

$$
65^{3/2} = (65^{1/2})^3 = (\sqrt{65})^3
$$

Frequently Asked Questions

What is the shape of the tank formed by rotating the curve $y=4x^2$ about the y-axis from $x=0$ to $x=1$?

The rotation of the parabola $y=4x^2$ about the y-axis from $x=0$ to $x=1$ creates a solid of revolution resembling a conical or paraboloidal tank, symmetric about the y-axis.

How can we determine the volume of water in the tank using the given curve $y=4x^2$?

By using the method of cylindrical shells or disks, integrating the cross-sectional area from $x=0$ to $x=1$, or applying the volume of revolution formula, we can compute the tank's volume based on the curve $y=4x^2$.

What is the significance of the density of water in calculating the total mass of water in the tank?

The density of water allows us to convert the volume of the tank into mass by multiplying volume with density, which is essential for understanding the weight and other physical properties of the contained water.

How do we set up the integral to find the volume of the tank formed by rotating $y=4x^2$ about the y-axis?

Using the shell method, the volume $V = 2\pi\int$ from $x=0$ to $x=1$ of $x\, y\, dx$, where $y=4x^2$, or converting to y in terms of x, and integrating accordingly, depending on the chosen method.

Why is understanding the shape and volume of the tank important in practical applications?

Knowing the shape and volume helps in designing efficient storage tanks, calculating the amount of water stored, assessing structural integrity, and managing resource allocation effectively.

Additional Resources

Water Density in a Rotated Tank: An In-Depth Analysis of a Rotationally Symmetric Water Container

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Introduction

In the realm of fluid mechanics and engineering design, understanding how liquids behave within various geometries is vital—whether for industrial applications, environmental modeling, or scientific research. Today, we delve into a fascinating scenario: a tank formed by rotating the curve $y = 4x^2$ about the y-axis, which is now filled with water. This exploration will not only analyze the physical structure of the tank but will also focus on the critical aspect of the water density within it, considering both theoretical calculations and practical implications.

The Geometric Foundation: Constructing the Tank

Understanding the Profile: $y = 4x^2$

The initial step involves examining the curve $y = 4x^2$, a classic parabola opening upward. This parabola serves as the generator of our tank's shape before rotation.

- Key features:
- Symmetric about the y-axis.
- Vertex at the origin (0,0).
- For any x, y is positive, indicating the parabola extends infinitely upward.

Revolution About the Y-Axis: Creating the 3D Shape

Rotating $y = 4x^2$ around the y-axis generates a solid of revolution with a distinctive, symmetrical shape—often termed a paraboloid of revolution.

- Method of construction:
- Each point (x, y) on the parabola traces a circle in 3D space with radius $r = |x|$.
- The resulting surface is a smooth, continuous shell that encloses a volume capable of holding water.

- Visualize:
- Imagine spinning the parabola around the y-axis; the shape resembles a glass or a bowl, tapering as it extends upward.

Mathematical Framework: Volume and Cross-Section Analysis

Defining the Limits of the Tank

- To specify the volume, we need to determine the bounds along the y-axis.
- Typically, the parabola extends from y=0 (at x=0) upward to some maximum y-value, say y = y_max, corresponding to a chosen x-value.

Suppose the tank is bounded between $y=0$ and $y=h$ (some height), which corresponds to x-values:

$$
x = \pm \frac{\sqrt{y}}{2}
$$

since $y = 4x^2 \Rightarrow x = \pm\sqrt{(y/4)}$.

Calculating the Volume of the Tank

Using the washer method or cylindrical shells, the volume V of the tank can be determined.

- Method: Shells

For a thin horizontal slice at height y, the shell's radius is $r = |x| = \sqrt{(y/4)}$, and the shell thickness is dy.

The volume element:

$$
dV = 2\pi r \times (\text{circumference}) \times (\text{height})
$$

Since the shells are formed by rotating x from $-\sqrt{(y/4)}$ to $\sqrt{(y/4)}$:

$$
dV = 2\pi r \times (\text{length of the shell}) \times dy
$$

But more straightforwardly, the volume can be computed as:

$$
V = \int_{0}^{h} \pi [x_{outer}^2 - x_{inner}^2] dy
$$

For the full paraboloid, x varies symmetrically, so the volume:

$$
V = \int_{0}^{h} \pi \left(\left(\frac{\sqrt{y}}{2}\right)^2 - 0^2\right) dy = \int_{0}^{h} \pi \frac{y}{4} dy
$$

$$
V = \frac{\pi}{4} \int_{0}^{h} y dy = \frac{\pi}{4} \left[\frac{y^2}{2}\right]_0^{h} = \frac{\pi}{8} h^2
$$

- Result:

$$
\boxed{
V = \frac{\pi}{8} h^2
}
$$

This formula indicates the volume depends quadratically on the height.

Water Filling the Tank: Density Considerations

Now, shifting focus from the geometric to the physical, we examine the water filled in this tank, with particular attention to water density.

What Is Water Density?

- Definition: Water density (ρ) is the mass of water per unit volume, typically expressed in kg/m^3.
- Standard value at room temperature (~20°C): Approximately 998 kg/m^3.
- Temperature dependence: Density varies with temperature; it increases slightly as temperature decreases and decreases with heating.

Implications of Density in the Context of the Tank

- Hydrostatic Pressure: The density of water directly influences the pressure exerted at the bottom of the tank.

$$
P = \rho g h
$$

where:
- P = pressure,
- ρ = density of water,
- g = acceleration due to gravity (~9.81 m/s^2),
- h = height of water column.

- Structural Design: Accurate knowledge of water density is essential for designing tank walls to withstand hydrostatic forces.

- Flow and Dynamics: Density impacts flow rates, buoyancy, and mixing within the tank.

Determining the Density of Water in the Tank

In most practical scenarios, the density of water is a known value, but for scientific accuracy or in specialized conditions, it can be measured or calculated.

Methods to Determine Water Density

1. Hydrometer Measurement:
- A simple device that floats at a specific level depending on water density.
- Calibration allows conversion from hydrometer reading to density.

2. Mass-Volume Method:
- Measure the mass of a known volume of water:

$$
\rho = \frac{m}{V}
$$

- Requires precise measurements of mass (using a scale) and volume (using calibrated containers).

3. Using Standard Conditions:
- At 20°C, assume 998 kg/m^3.
- For other temperatures, consult water density tables or equations.

Practical Factors Affecting Water Density in the Tank

- Temperature Variations: Changes in temperature alter density; colder water is denser.
- Impurities and Salinity: Saltwater has higher density (~1025 kg/m^3 at 20°C).
- Pressure Effects: At typical atmospheric pressures, the effect on density is negligible, but high-pressure environments can influence it.

Significance in Engineering and Scientific Applications

Understanding water density within this rotationally symmetric tank is more than an academic exercise; it informs various practical tasks:

Structural Integrity and Safety

- Ensuring the tank's walls withstand hydrostatic pressures based on water density.
- Calculating the total force on the tank walls:

$$
F = \int_{0}^{h} \rho g y \times \text{area} \, dy
$$

- Properly accounting for water density ensures safety margins are adequate.

Fluid Dynamics and Mixing

- Density gradients can cause stratification or mixing issues.
- Accurate density measurements help simulate flow patterns, especially if temperature or salinity varies.

Environmental and Scientific Research

- Modeling water behavior in natural or artificial reservoirs.
- Designing tanks for experiments where precise water properties are critical.

Summary and Final Thoughts

The tank formed by rotating $y=4x^2$ about the y-axis presents a compelling case study in geometric design and physical properties interplay. Its volume, derived via integral calculus, is proportional to the square of its height, emphasizing the importance of precise measurements in both construction and scientific analysis.

Water density, a fundamental physical property, influences structural design, safety considerations, and fluid behavior within the tank. While standard values provide a baseline for calculations, real-world conditions such as temperature, salinity, and pressure can modify the actual density, necessitating careful measurement or adjustment.

In conclusion, understanding the density of water in such a uniquely shaped tank is essential for ensuring its safe operation, optimizing its design, and accurately modeling fluid behavior. Whether for industrial storage, scientific research, or environmental management, integrating geometric insights with physical properties like water density remains a cornerstone of engineering excellence.

Final Note: For practitioners and researchers working with complex geometries like the paraboloid of revolution, combining calculus, material science, and fluid mechanics knowledge is vital. This comprehensive analysis underscores the importance of interdisciplinary understanding in creating safe, efficient, and scientifically accurate fluid storage solutions.

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