Calculate The Circulation Of The Field F Around The Closed Curve C. F = X2y3 I +x2y3 J; Curve C Is The

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Introduction

Calculating the circulation of a vector field around a closed curve is a fundamental concept in vector calculus, with widespread applications in physics, engineering, and mathematics. The circulation provides insights into the rotational behavior of the field and helps in understanding phenomena such as fluid flow, electromagnetic fields, and more. In this article, we will explore how to compute the circulation of the vector field F = $x^2 y^3$ i + $x^2 y^3$ j around a specific closed curve C.

Understanding the Vector Field F

Definition of the Field F

The vector field F is given as:

$$
F(x, y) = x^2 y^3 \mathbf{i} + x^2 y^3 \mathbf{j}
$$

which can be expressed component-wise as:

$$
F(x, y) = (P, Q) = (x^2 y^3, x^2 y^3)
$$

This indicates that both the x and y components are identical functions of x and y.

Properties of the Field

- Symmetry: The field has symmetric components in i and j, simplifying certain calculations.
- Differentiability: The functions involved are polynomial and infinitely differentiable, which makes the field smooth and suitable for applying calculus theorems such as Green's theorem.
- Behavior: The field's magnitude increases with larger x and y values, especially in regions where both are positive.

Understanding the Curve C

Nature of the Curve

The problem statement indicates that C is a closed curve. Typically, in such problems, C can be a circle, ellipse, or any other simple closed curve. For clarity and illustrative purposes, we'll consider C as a circle centered at the origin with radius r, i.e.,

$$
C: x^2 + y^2 = r^2
$$

This choice simplifies calculations and is common in vector calculus problems.

Parameterization of C

To compute the circulation, we need a parameterization of C:

$$
\mathbf{r}(t) = r \cos t\, \mathbf{i} + r \sin t\, \mathbf{j}, \quad t \in [0, 2\pi]
$$

Differential elements:

$$
d\mathbf{r} = \frac{d\mathbf{r}}{dt} dt = (-r \sin t\, \mathbf{i} + r \cos t\, \mathbf{j}) dt
$$

Calculating the Circulation of F Around C

Definition of Circulation

The circulation Γ of a vector field F around a closed curve C is given by the line integral:

$$
\Gamma = \oint_{C} \mathbf{F} \cdot d\mathbf{r}
$$

which expands to:

$$
\Gamma = \oint_{C} P\, dx + Q\, dy
$$

where dx and dy are differentials along the parameterized curve.

Applying Green's Theorem

Green's theorem relates the line integral around C to a double integral over the region D enclosed by C:

$$
\oint_{C} P dx + Q dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy
$$

This approach simplifies calculations, especially when C is a simple, closed curve like a circle.

Calculating the Partial Derivatives

Given:

$$
P = x^2 y^3, \quad Q = x^2 y^3
$$

Compute:

$$
\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} (x^2 y^3) = 2x y^3
$$

$$
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} (x^2 y^3) = 3 x^2 y^2
$$

Therefore,

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 x y^3 - 3 x^2 y^2
$$

Evaluating the Double Integral

Region D: The Disk $x^2 + y^2 \leq r^2$

Since C is a circle of radius r, the region D is the disk:

$$
D = \{ (x, y) \mid x^2 + y^2 \leq r^2 \}
$$

Switching to Polar Coordinates

Using polar coordinates:

$$
x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad \rho \in [0, r], \quad \theta \in [0, 2\pi]
$$

The Jacobian determinant:

$$
dx\, dy = \rho\, d\rho\, d\theta
$$

Express the integrand in polar coordinates:

$$
2 x y^3 - 3 x^2 y^2
$$

Substitute:

$$
x = \rho \cos \theta, \quad y = \rho \sin \theta
$$

Compute each term:

$$
2 x y^3 = 2 (\rho \cos \theta) (\rho \sin \theta)^3 = 2 \rho \cos \theta \times \rho^3 \sin^3 \theta = 2 \rho^4 \cos \theta \sin^3 \theta
$$

$$
3 x^2 y^2 = 3 (\rho \cos \theta)^2 (\rho \sin \theta)^2 = 3 \rho^2 \cos^2 \theta \times \rho^2 \sin^2 \theta = 3 \rho^4 \cos^2 \theta \sin^2 \theta
$$

The integrand becomes:

$$
2 \rho^4 \cos \theta \sin^3 \theta - 3 \rho^4 \cos^2 \theta \sin^2 \theta
$$

The double integral:

$$
\iint_D \left(... \right) dx\, dy = \int_0^{2\pi} \int_0^r \left(2 \rho^4 \cos \theta \sin^3 \theta - 3 \rho^4 \cos^2 \theta \sin^2 \theta \right) \rho\, d\rho\, d\theta
$$

Simplify:

$$
= \int_0^{2\pi} \int_0^r \left(2 \rho^5 \cos \theta \sin^3 \theta - 3 \rho^5 \cos^2 \theta \sin^2 \theta \right) d\rho\, d\theta
$$

Solving the Integral

Integral over p

$$
\int_0^R \rho^5 d\rho = \frac{R^6}{6}
$$

Thus, the expression becomes:

$$
\frac{R^6}{6} \int_0^{2\pi} \left(2 \cos \theta \sin^3 \theta - 3 \cos^2 \theta \sin^2 \theta \right) d\theta
$$

Now, focus on the angular integral:

$$
I = \int_0^{2\pi} \left(2 \cos \theta \sin^3 \theta - 3 \cos^2 \theta \sin^2 \theta \right) d\theta
$$

Break into two integrals:

$$
I = 2 \int_0^{2\pi} \cos \theta \sin^3 \theta d\theta - 3 \int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta
$$

Evaluating the Angular Integrals

First integral:

\[

I_1 = \int_0^{2\pi} \cos \theta \sin^3 \theta d\theta
\]

Use substitution:

\[
u = \sin \theta \Rightarrow du = \cos \theta d\theta
\]

When \(\theta = 0\), \(u=0\); when \(\theta=2 \pi\), \(u=0\) (since sine is periodic). The integral over a period:

\[
I_1 = \int_{u=0}^{0} u^3 du
\]

which is zero, as the limits are the same. But to be more precise, note that the integral over a full period cancels out due to symmetry.

Alternatively, note that \(\cos \theta\) is symmetric and

Frequently Asked Questions

How do you compute the circulation of the vector field $F = x^2y^3 i + x^2y^3 j$ around a closed curve C?

To compute the circulation, you evaluate the line integral of F along C: $\oint_C F \cdot dr$. Depending on the curve, you may parametrize C and compute the integral directly or apply Green's theorem to convert it into a double integral over the region enclosed by C.

What role does Green's theorem play in calculating the circulation of $F = x^2y^3 i + x^2y^3 j$ around a closed curve C?

Green's theorem relates the line integral around a closed curve C to a double integral over the region D enclosed by C. For this field, it simplifies the calculation by converting the circulation into a double integral of $(\partial Q/\partial x - \partial P/\partial y)$ over D, where P and Q are the components of F.

Is the vector field $F = x^2y^3\, i + x^2y^3\, j$ conservative, and how does that affect the circulation around closed curves?

Since the components P and Q of F are both x^2y^3, their mixed partial derivatives are equal $(\partial Q/\partial x = \partial P/\partial y)$. This indicates F is conservative in simply connected regions, meaning the circulation around any closed curve C is zero.

How does the shape of curve C influence the calculation of the circulation for the given field?

The shape and position of curve C determine the region D over which Green's theorem applies. For simple shapes like circles or rectangles, the double integral is easier to evaluate; for complex curves, parametrization or numerical methods might be necessary.

Can the circulation of $F = x^2y^3\, i + x^2y^3\, j$ be zero, and under what conditions?

Yes, if F is conservative or the enclosed region has symmetry leading to cancellation, the circulation around C can be zero. Specifically, if the curl of F is zero or the integral over the region cancels out, the circulation will be zero.

Additional Resources

Calculating the Circulation of a Vector Field Around a Closed Curve: An Expert Analysis

In the realm of vector calculus, understanding how a vector field behaves around closed curves is fundamental, especially in applications spanning fluid dynamics, electromagnetism, and engineering. A key concept in this domain is circulation, which quantifies the tendency of a vector field to "circulate" around a given loop. This article provides an in-depth, step-by-step examination of calculating the circulation of a specific vector field, $\mathbf{F} = x^2 y^3 \mathbf{i} + x^2 y^3 \mathbf{j}$, around a closed curve C. We aim to dissect the problem thoroughly, offering detailed insights while maintaining clarity for both students and practitioners.

Understanding the Problem: Vector Field and Curve Description

Before delving into calculations, it is essential to understand the components involved:

The Vector Field $\mathbf{F}$

The given vector field:

$$
\mathbf{F}(x, y) = x^2 y^3 \mathbf{i} + x^2 y^3 \mathbf{j}
$$

can be interpreted as a two-dimensional field where both components are identical functions $x^2 y^3$. Such symmetry suggests interesting circulation behavior, especially around symmetric curves.

The Curve C

The problem statement mentions "Curve C Is The," but appears incomplete. Commonly in vector calculus problems, C is a simple, closed, and smooth curve such as:

- The circle $x^2 + y^2 = R^2$
- The square with vertices at specific points
- An ellipse or other symmetric boundary

For this analysis, we'll consider the most typical and illustrative case: the circle of radius R centered at the origin:

$$
C: x^2 + y^2 = R^2
$$

This choice simplifies calculations while exemplifying key concepts. If the original problem intended a different curve, the methodology remains similar, with adjustments for the specific parameterization.

Fundamentals of Circulation and Its Calculation

What is Circulation?

In vector calculus, circulation of a vector field $\mathbf{F}$ around a closed curve C is defined as:

$$
\text{Circulation} = \oint_C \mathbf{F} \cdot d\mathbf{r}
$$

where:

- $\oint_C$ indicates a line integral over the closed curve C
- $d\mathbf{r}$ is an infinitesimal displacement vector along C

Physically, circulation measures the total "rotation" or "twist" induced by $\mathbf{F}$ around C.

Methods to Compute Circulation

Two primary methods are employed:

1. Direct Line Integral Calculation: Parameterize C and evaluate the

integral explicitly.
2. Use of Green's Theorem: Convert the line integral into a double integral over the region D enclosed by C, which simplifies calculations in many cases.

Given the symmetry and nature of $\mathbf{F}$, Green's theorem provides an efficient pathway.

Applying Green's Theorem: Transforming the Line Integral

Green's Theorem Overview

Green's theorem relates a line integral around a simple closed curve C to a double integral over the region D it encloses:

$$
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy
$$

where:

- $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$
- $P = x^2 y^3$
- $Q = x^2 y^3$

This theorem holds for positively oriented, simple, closed curves that are piecewise smooth.

Calculating the Curl (or the integrand)

The expression inside the double integral:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}
$$

\]

becomes the key to simplifying the circulation calculation.

Step-by-Step Calculation of Circulation

Step 1: Compute the Partial Derivatives

Given:

$$
P = x^2 y^3, \quad Q = x^2 y^3
$$

Calculate:

$$
\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} (x^2 y^3) = 2x y^3
$$

$$
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} (x^2 y^3) = 3 x^2 y^2
$$

Step 2: Formulate the Integrand

Subtract:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x y^3 - 3 x^2 y^2
$$

This is the integrand for Green's theorem.

Step 3: Define the Region D

Assuming C is the circle $x^2 + y^2 = R^2$, the region:

$$
D: x^2 + y^2 \le R^2
$$

is a disk centered at the origin.

Step 4: Switch to Polar Coordinates

To evaluate the double integral:

$$
\iint_D (2x y^3 - 3 x^2 y^2) dx dy
$$

convert to polar coordinates:

$$
x = r \cos \theta, \quad y = r \sin \theta
$$

and the Jacobian:

$$
dx dy = r dr d\theta
$$

Express the integrand in terms of r and θ:

$$
2x y^3 = 2 (r \cos \theta) (r \sin \theta)^3 = 2 r \cos \theta \cdot r^3 \sin^3 \theta = 2 r^4 \cos \theta \sin^3 \theta
$$

$$
-3 x^2 y^2 = -3 (r \cos \theta)^2 (r \sin \theta)^2 = -3 r^2 \cos^2 \theta \cdot r^2 \sin^2 \theta = -3 r^4 \cos^2 \theta \sin^2 \theta
$$

Thus, the integrand becomes:

$$
r^4 \left[2 \cos \theta \sin^3 \theta - 3 \cos^2 \theta \sin^2 \theta \right]
$$

The double integral:

$$
\int_0^{2\pi} \int_0^R r^4 \left[2 \cos \theta \sin^3 \theta - 3 \cos^2 \theta \sin^2 \theta \right] r \, dr \, d\theta
$$

Simplifies to:

$$
\int_0^{2\pi} \left[\int_0^R r^5 \, dr \right] \left[2 \cos \theta \sin^3 \theta - 3 \cos^2 \theta \sin^2 \theta \right] d\theta
$$

Step 5: Integrate over r

$$
\int_0^R r^5 \, dr = \frac{R^6}{6}
$$

This yields:

$$
\frac{R^6}{6} \int_0^{2\pi} \left[2 \cos \theta \sin^3 \theta - 3 \cos^2 \theta \sin^2 \theta \right] d\theta
$$

Step 6: Evaluate the Angular Integral

Define:

$$
I = \int_0^{2\pi} \left[2 \cos \theta \sin^3 \theta - 3 \cos^2 \theta \sin^2 \theta \right] d\theta
$$

Break into two integrals:

$$
I = 2 \int_0^{2\pi} \cos \theta \sin^3 \theta d\theta - 3 \int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta
$$

Integral 1: $\int_0^{2\pi} \cos \theta \sin^3 \theta d\theta$

Use substitution:

$$
u = \sin \theta \implies du = \cos \theta d\theta
$$

When $\theta = 0$, $u=0$; when $\theta = 2\pi$, $u=0$. Since u is periodic over 2π, the integral over the full period:

$$
\int_0^{2\pi} \cos \theta \sin^3 \theta d\theta = \int_{u=0}^{0} u^3 du
$$

which

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