Consider The Initial Value Problem Y" + Y' + Y = 0, Y(0) = 3, Y'(0) = K, Where , And K Are Constants.

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Understanding and solving initial value problems (IVPs) for differential equations is fundamental in applied mathematics, physics, engineering, and many scientific disciplines. In this article, we focus on a specific second-order linear homogeneous differential equation with constant coefficients:

$$Y'' + Y' + Y = 0, \quad Y(0) = 3, \quad Y'(0) = K$$

where K is an arbitrary constant. We will explore the process of solving this problem step-by-step, interpret the solution's implications, and discuss how the initial conditions influence the behavior of the solution.

Understanding the Differential Equation $Y'' + Y' + Y = 0$

Nature of the Equation

- It is a second-order linear homogeneous differential equation with constant coefficients.
- The general form is:

$$aY'' + bY' + cY = 0$$

with $a = 1$, $b = 1$, and $c = 1$.

- Such equations appear frequently in modeling systems like damped harmonic oscillators, electrical circuits, and mechanical vibrations.

Significance of the Equation

- The equation models systems experiencing damping, where the term Y' represents damping effects.
- The behavior of solutions—whether oscillatory, exponential decay, or growth—depends on the roots of the characteristic equation.

Solving the Differential Equation

Step 1: Formulate the Characteristic Equation

- To solve, assume a solution of the form $Y(t) = e^{rt}$,
- Substituting into the differential equation yields:

$$r^2 e^{rt} + r e^{rt} + e^{rt} = 0$$

- Dividing through by e^{rt} (which is never zero):

$$r^2 + r + 1 = 0$$

- This quadratic equation is called the characteristic equation.

Step 2: Find the Roots of the Characteristic Equation

- The roots are given by:

$$r = \frac{ -b \pm \sqrt{b^2 - 4ac} }{2a}$$

Substituting $a=1$, $b=1$, $c=1$:

$$r = \frac{ -1 \pm \sqrt{1^2 - 4 \times 1 \times 1} }{2} = \frac{ -1 \pm \sqrt{1 - 4} }{2} = \frac{ -1 \pm \sqrt{-3} }{2}$$

- Simplify the square root:

$$r = \frac{ -1 \pm i \sqrt{3} }{2}$$

- The roots are complex conjugates:

$$r_1 = \frac{ -1 }{2} + i \frac{ \sqrt{3} }{2}$$
$$r_2 = \frac{ -1 }{2} - i \frac{ \sqrt{3} }{2}$$

Step 3: Write the General Solution

- For complex roots $\alpha \pm i \beta$, the general solution is:

$$Y(t) = e^{\alpha t} \left(C_1 \cos \beta t + C_2 \sin \beta t \right)$$

- Here, $\alpha = -\frac{1}{2}$, $\beta = \frac{\sqrt{3}}{2}$.

- Therefore:

$$Y(t) = e^{ -\frac{1}{2} t } \left(C_1 \cos \frac{\sqrt{3}}{2} t + C_2 \sin \frac{\sqrt{3}}{2} t \right)$$

Applying Initial Conditions to Find Constants

Initial Conditions Given

- $Y(0) = 3$
- $Y'(0) = K$

Step 1: Find $Y(0)$

- Substitute $t=0$:

$$Y(0) = e^{0} (C_1 \cos 0 + C_2 \sin 0) = C_1 \times 1 + C_2 \times 0 = C_1$$

- Since $Y(0) = 3$:

$$C_1 = 3$$

Step 2: Find $Y'(t)$ and apply $t=0$

- Differentiate $Y(t)$:

\[

Y'(t) = \frac{d}{dt} \left[e^{ -\frac{1}{2} t } (C_1 \cos \frac{\sqrt{3}}{2} t + C_2 \sin \frac{\sqrt{3}}{2} t) \right]
\]

- Use the product rule:

\[
Y'(t) = e^{ -\frac{1}{2} t } \left[-\frac{1}{2} (C_1 \cos \frac{\sqrt{3}}{2} t + C_2 \sin \frac{\sqrt{3}}{2} t) + C_1 \left(- \frac{\sqrt{3}}{2} \sin \frac{\sqrt{3}}{2} t \right) + C_2 \left(\frac{\sqrt{3}}{2} \cos \frac{\sqrt{3}}{2} t \right) \right]
\]

- Evaluate at \(t=0 \):

\[
Y'(0) = e^{0} \left[-\frac{1}{2} (C_1 \times 1 + C_2 \times 0) + C_1 \left(0 \right) + C_2 \left(\frac{\sqrt{3}}{2} \times 1 \right) \right]
\]

Simplifies to:

\[
Y'(0) = -\frac{1}{2} C_1 + \frac{\sqrt{3}}{2} C_2
\]

- Recall \(C_1 = 3 \):

\[
Y'(0) = -\frac{1}{2} \times 3 + \frac{\sqrt{3}}{2} C_2 = -\frac{3}{2} + \frac{\sqrt{3}}{2} C_2
\]

- Given \(Y'(0) = K \), solve for \(C_2 \):

\[
K = -\frac{3}{2} + \frac{\sqrt{3}}{2} C_2
\]

\[
\Rightarrow \frac{\sqrt{3}}{2} C_2 = K + \frac{3}{2}
\]

\[
\Rightarrow C_2 = \frac{2}{\sqrt{3}} \left(K + \frac{3}{2} \right)

\]

Final Solution of the Initial Value Problem

Putting everything together, the solution is:

$$
Y(t) = e^{ -\frac{1}{2} t } \left[3 \cos \frac{\sqrt{3}}{2} t + \frac{2}{\sqrt{3}} \left(K + \frac{3}{2} \right) \sin \frac{\sqrt{3}}{2} t \right]
$$

This formula explicitly shows how the initial conditions $Y(0)$ and $Y'(0)$ influence the solution through the constants C_1 and C_2.

Interpreting the Solution and Its Behavior

Behavior Based on Constants K

- The initial velocity K determines the amplitude and phase of oscillations.
- For different values of K:

  - It affects the magnitude of the sine term, influencing the initial slope of the solution.
  - If $K = -\frac{3}{2}$, then $C_2 = 0$, simplifying the solution to a purely exponential decay without oscillation.

Nature of the Solution

- Since roots are complex conjugates with negative real parts ($-\frac{1}{2}$), the solution exhibits

damped oscillations.
- The exponential term $e^{ -\frac{1}{2} t }$ causes the oscillations to decay over time.
- The frequency of oscillation is $\frac{\sqrt{3}}{2}$, related to the

Frequently Asked Questions

What is the initial value problem given for the differential equation Y" + Y' + Y = 0?

The initial value problem is $Y'' + Y' + Y = 0$, with initial conditions $Y(0) = 3$ and $Y'(0) = K$, where K is a constant.

How do you find the general solution of the differential equation Y" + Y' + Y = 0?

Solve the characteristic equation $r^2 + r + 1 = 0$, then find roots and write the general solution based on those roots.

What are the roots of the characteristic equation for Y" + Y' + Y = 0?

The roots are complex: $r = -1/2 \pm i\sqrt{3}/2$, indicating a damped oscillatory solution.

What is the explicit form of the general solution for the differential equation?

$Y(t) = e^{(-t/2)} [A \cos(\sqrt{3}\, t/2) + B \sin(\sqrt{3}\, t/2)]$, where A and B are constants determined by initial conditions.

How do you determine the constants A and B using the initial conditions Y(0) = 3 and Y'(0) = K?

Substitute $t=0$ into the general solution and its derivative to create equations, then solve for A and B using the given initial conditions.

What is the value of constant A given Y(0) = 3?

At $t=0$, $Y(0) = A = 3$, since $e^0 = 1$ and $\cos(0) = 1$, $\sin(0) = 0$.

How do you find the constant B using Y'(0) = K?

Differentiate the general solution, evaluate at t=0 to find Y'(0), then solve for B using the initial condition Y'(0) = K.

What is the expression for Y'(t) for the solution?

$Y'(t) = e^{-t/2} [-A/2 \cos(\sqrt{3}\, t/2) - (\sqrt{3}/2)A \sin(\sqrt{3}\, t/2) + B/2 \sin(\sqrt{3}\, t/2) + (\sqrt{3}/2) B \cos(\sqrt{3}\, t/2)]$.

How does the value of K affect the particular solution of the differential equation?

The value of K determines B in the solution, thus shifting the solution's phase and amplitude according to the initial velocity condition.

What is the significance of the complex roots in the solution to this differential equation?

The complex roots indicate that the solution exhibits oscillatory behavior with exponential decay, characteristic of damped harmonic motion.

Additional Resources

Consider The Initial Value Problem Y'' + Y' + Y = 0, Y(0) = 3, Y'(0) = K, Where K Are Constants is a classic example of a second-order linear differential equation with constant coefficients. Such problems are fundamental in understanding various physical systems, including mechanical vibrations, electrical circuits, and control systems. Analyzing this initial value problem (IVP) involves determining the general solution to the differential equation and then applying the initial conditions to find the specific solution. This guide walks you through the process step-by-step, providing insights into the mathematical techniques involved and the implications of different values for the constant K.

Understanding the Differential Equation: Y'' + Y' + Y = 0

The differential equation in question,

$$Y'' + Y' + Y = 0,$$

is a homogeneous linear differential equation with constant coefficients. Here, Y is a function of an independent variable (usually time t), and primes denote derivatives with respect to this variable.

This type of equation typically models systems where the future state depends linearly on the current and past states, such as damped harmonic oscillators or RLC circuits.

The Initial Conditions

The problem provides initial conditions:

$$ Y(0) = 3, \quad Y'(0) = K, $$

where 3 and K are constants. These initial conditions specify the initial position and velocity (or analogous quantities depending on the physical context) of the system at $t=0$.

Step 1: Formulate the Characteristic Equation

To solve the differential equation, we start by proposing a solution of the form:

$$ Y(t) = e^{rt}, $$

where r is a constant to be determined. Substituting into the differential equation:

$$
r^2 e^{rt} + r e^{rt} + e^{rt} = 0,
$$

which simplifies (since $e^{rt} \neq 0$) to the characteristic equation:

$$
r^2 + r + 1 = 0.
$$

Step 2: Solve the Characteristic Equation

The roots of the quadratic are:

$$
r = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.
$$

Since the discriminant is negative (-3), the roots are complex conjugates:

$$
r = \frac{-1}{2} \pm i \frac{\sqrt{3}}{2}.
$$

Expressed explicitly:

$$
r = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}.
$$

Step 3: Write the General Solution

For complex roots $\alpha \pm i \beta$, the general solution is:

$$
Y(t) = e^{\alpha t} \left(C_1 \cos \beta t + C_2 \sin \beta t \right),
$$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

Plugging in $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$, the general solution becomes:

$$
Y(t) = e^{-\frac{1}{2} t} \left(C_1 \cos \frac{\sqrt{3}}{2} t + C_2 \sin \frac{\sqrt{3}}{2} t \right).
$$

Step 4: Apply Initial Conditions to Find Particular Solution

Initial Condition 1: $Y(0) = 3$

At $t=0$:

$$
Y(0) = e^{0} \left(C_1 \cos 0 + C_2 \sin 0 \right) = C_1 \times 1 + C_2 \times 0 = C_1.
$$

Thus:

$$
C_1 = 3.
$$

Initial Condition 2: $Y'(0) = K$

To find C_2, we need $Y'(t)$. Differentiating $Y(t)$:

$$
Y(t) = e^{-\frac{1}{2} t} \left(C_1 \cos \frac{\sqrt{3}}{2} t + C_2 \sin \frac{\sqrt{3}}{2} t \right).
$$

Applying the product rule:

$$
Y'(t) = e^{-\frac{1}{2} t} \left[-\frac{1}{2} \left(C_1 \cos \frac{\sqrt{3}}{2} t + C_2 \sin \frac{\sqrt{3}}{2} t \right) + \left(C_1 \left(- \frac{\sqrt{3}}{2} \sin \frac{\sqrt{3}}{2} t \right) + C_2 \left(\frac{\sqrt{3}}{2} \cos \frac{\sqrt{3}}{2} t \right) \right) \right].
$$

At $t=0$:

$$
Y'(0) = \left[-\frac{1}{2} (C_1 \times 1 + C_2 \times 0) + (C_1 \times 0 + C_2 \times \frac{\sqrt{3}}{2} \times 1) \right] \times e^{0} = -\frac{1}{2} C_1 + \frac{\sqrt{3}}{2} C_2.
$$

Recall $C_1 = 3$, so:

$$
Y'(0) = -\frac{1}{2} \times 3 + \frac{\sqrt{3}}{2} C_2 = -\frac{3}{2} + \frac{\sqrt{3}}{2} C_2.
$$

Set equal to K:

$$
K = -\frac{3}{2} + \frac{\sqrt{3}}{2} C_2,
$$

which yields:

$$
\frac{\sqrt{3}}{2} C_2 = K + \frac{3}{2},
$$

and

$$
C_2 = \frac{2}{\sqrt{3}} \left(K + \frac{3}{2} \right) = \frac{2}{\sqrt{3}} K + \frac{2}{\sqrt{3}} \times \frac{3}{2} = \frac{2}{\sqrt{3}} K + \frac{3}{\sqrt{3}}.
$$

Simplify:

$$
C_2 = \frac{2}{\sqrt{3}} K + \sqrt{3}.
$$

Final Solution

Putting everything together, the particular solution to the initial value problem is:

$$
Y(t) = e^{-\frac{1}{2} t} \left[3 \cos \frac{\sqrt{3}}{2} t + \left(\frac{2}{\sqrt{3}} K + \sqrt{3} \right) \sin \frac{\sqrt{3}}{2} t \right].
$$

This expression captures the system's behavior for any initial velocity K. The solution elegantly demonstrates how initial conditions influence the amplitude and phase of the oscillations.

Analyzing the Behavior of the Solution

Impact of the Constant K

The value of K significantly affects the initial velocity and consequently the system's response:

- If $K = -\frac{3}{2}$: The coefficient C_2 becomes zero, simplifying the solution to a purely cosine-based oscillation with exponential decay.
- If K increases: The sine component's amplitude increases, leading to a phase shift and larger initial velocity.
- If K decreases or is negative: The initial velocity is negative, affecting the initial direction of the oscillation.

Damped Oscillations

Because the exponential term $e^{-\frac{1}{2} t}$ involves a negative exponent, the solution exhibits damped oscillations that decay to zero as $t \to \infty$. The damping factor $\frac{1}{2}$ indicates how quickly the amplitude diminishes over time.

Oscillation Frequency

The oscillation frequency is governed by $\beta = \frac{\sqrt{3}}{2}$, corresponding to a period:

$$
T = \frac{2\pi}{\beta} = \frac{2\pi}{\frac{\sqrt{3}}{2}} = \frac{4\pi}{\sqrt{3}}.
$$

This periodicity reflects the natural frequency of the system influenced by the coefficients in the differential equation.

Special Cases and Further Analysis

Critically Damped and Overdamped Scenarios

While this specific problem is overdamped (due to complex roots), similar equations with different coefficients can lead to:

- Critical damping: Repeated real roots, resulting in solutions involving exponential decay and polynomial factors.
- Overdamped: Distinct real roots,

[Consider The Initial Value Problem Y Y Y 0 Y 0 3 Y 0 K Where And K Are Constants]()

Consider The Initial Value Problem Y Y Y 0 Y 0 3 Y 0 K Where And K Are Constants

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