Consider The Solid Bounded By Surfaces Given By Z=x2 + Y2, Z = 4, And Within The Cylinder X2 + Y2 = 1.

Consider The Solid Bounded By Surfaces Given By Z=x2 + Y2, Z = 4, And Within The Cylinder X2 + Y2 = 1.

Understanding the geometry and volume of this particular solid is a fundamental problem in multivariable calculus, with applications spanning physics, engineering, and mathematical modeling. This article provides a comprehensive exploration of the solid bounded by the paraboloid $Z = x^2 + y^2$, the plane $Z = 4$, and the cylindrical surface $x^2 + y^2 = 1$. We will analyze its description, visualization, mathematical formulation, and methods to compute its volume.

Understanding the Geometric Components

Before delving into calculations, it is essential to understand each surface that bounds the solid.

The Paraboloid: $Z = x^2 + y^2$

- The paraboloid opens upward, with its vertex at the origin ($0, 0, 0$).
- In the xy-plane, it is circularly symmetric, with the radius corresponding to a given height Z.
- As $x^2 + y^2$ increases, the surface rises quadratically.

The Plane: $Z = 4$

- A horizontal plane parallel to the xy-plane.
- It intersects the paraboloid at the points where $x^2 + y^2 = 4$.
- Since the paraboloid $Z = x^2 + y^2$ equals 4 at the circle $x^2 + y^2 = 4$, the paraboloid reaches the plane at its outer boundary.

The Cylinder: $x^2 + y^2 = 1$

- A circular cylinder aligned along the z-axis, with radius 1.
- It creates a cylindrical boundary that intersects the paraboloid and the plane.

Visualizing the Solid

The solid is the region enclosed by the paraboloid $Z = x^2 + y^2$, truncated above by the plane $Z=4$, and confined within the cylinder $x^2 + y^2 = 1$.

Key observations:

- The cylinder restricts the xy-region to $x^2 + y^2 \leq 1$.
- The paraboloid opens upward, with its lowest point at the origin.
- The plane $Z=4$ intersects the paraboloid along the circle $x^2 + y^2 = 4$, but since the cylinder restricts $x^2 + y^2 \leq 1$, the actual intersection within the cylinder occurs where $Z = x^2 + y^2 \leq 1$, which is below the plane $Z=4$.

Thus, the portion of the paraboloid within the cylinder extends from the vertex at $Z=0$ up to $Z=1$

), since at $x^2 + y^2 = 1$, $Z = 1$.

Determining the Region of Integration

To compute the volume of this solid, it is most convenient to switch to cylindrical coordinates because of the symmetry.

Cylindrical Coordinates

Recall the transformation:

$$
x = r \cos \theta \\
y = r \sin \theta \\
z = z
$$

where $r \geq 0$, $0 \leq \theta < 2\pi$, and z varies between the surfaces.

Surfaces in cylindrical coordinates:

- $Z = r^2$ (paraboloid)
- $Z = 4$ (plane)
- $r = 1$ (cylinder boundary)

Region in the xy-plane:

- $0 \leq r \leq 1$
- $0 \leq \theta < 2\pi$

Vertical bounds:

- For each fixed r, z ranges from the paraboloid $z = r^2$ up to the plane $z=4$.

Formulating the Volume Integral

The volume V of the solid can be computed using a triple integral over the region R:

$$
V = \iiint_{R} dV
$$

In cylindrical coordinates:

$$
dV = r \, dr \, d\theta \, dz
$$

Given the bounds:

- $0 \leq r \leq 1$
- $0 \leq \theta \leq 2\pi$
- z ranges from r^2 up to 4

the integral becomes:

$$
V = \int_{0}^{2\pi} \int_{0}^{1} \int_{z = r^2}^{z=4} r \, dz \, dr \, d\theta
$$

Calculating the Volume

Let's evaluate the triple integral step-by-step.

Step 1: Integrate with respect to z:

$$
\int_{z = r^2}^{4} r \, dz = r (4 - r^2)
$$

Step 2: Integrate with respect to r:

$$
\int_{0}^{1} r (4 - r^2) dr
$$

Break into two integrals:

$$
4 \int_{0}^{1} r \, dr - \int_{0}^{1} r^3 \, dr
$$

Calculating each:

- \(\int_{0}^{1} r \, dr = \frac{1}{2}\)

- \(\int_{0}^{1} r^3 \, dr = \frac{1}{4}\)

Plugging back:

\[

4 \times \frac{1}{2} - \frac{1}{4} = 2 - \frac{1}{4} = \frac{8}{4} - \frac{1}{4} = \frac{7}{4}

\]

Step 3: Integrate with respect to \(\theta \):

\[

\int_{0}^{2\pi} d\theta = 2\pi

\]

Final volume:

\[

V = 2\pi \times \frac{7}{4} = \frac{14 \pi}{4} = \frac{7 \pi}{2}

\]

Result and Interpretation

The volume of the solid bounded by the paraboloid \(Z = x^2 + y^2 \), the plane \(Z=4 \), and within the cylinder \(x^2 + y^2 = 1 \) is:

\[

\boxed{

V = \frac{7 \pi}{2}

}

\]

This elegant result demonstrates the interplay between the symmetry of the surfaces and the power of cylindrical coordinates in simplifying volume calculations.

Applications of This Geometric Analysis

Understanding such geometric shapes and their volumes is fundamental in various scientific and engineering fields. Some applications include:

- Design of optical systems: Calculating the volume of curved surfaces.
- Fluid dynamics: Determining the volume of fluid within bounded regions.
- Structural engineering: Analyzing the material volume of components with curved boundaries.
- Mathematical modeling: Developing intuition about three-dimensional shapes bounded by multiple surfaces.

Extensions and Related Problems

This analysis opens avenues for exploring more complex volume calculations:

- Varying the bounds: Changing the plane's height or the cylinder's radius to analyze different bounded solids.

- Other surfaces: Replacing the paraboloid with hyperboloids or ellipsoids.
- Surface integrals: Calculating surface areas of the bounding surfaces.

Conclusion

The problem of determining the volume of the solid bounded by $Z = x^2 + y^2$, $Z=4$, and within the cylinder $x^2 + y^2 = 1$ showcases fundamental techniques in multivariable calculus. By leveraging symmetry and cylindrical coordinates, the volume calculation becomes straightforward, yielding the result $\frac{7 \pi}{2}$. Such problems deepen our understanding of three-dimensional geometry and the power of integral calculus in solving real-world spatial problems.

Frequently Asked Questions

What is the solid bounded by the surfaces $Z = x^2 + y^2$, $Z = 4$, and the cylinder $x^2 + y^2 = 1$?

The solid is the region enclosed between the paraboloid $Z = x^2 + y^2$ and the horizontal plane $Z = 4$, confined within the cylinder $x^2 + y^2 = 1$.

How can we set up the bounds for integrating over this solid in cylindrical coordinates?

In cylindrical coordinates, r varies from 0 to 1, θ from 0 to 2π, and z from r^2 (the paraboloid) up to 4 (the plane).

What is the volume of the solid bounded by these surfaces?

The volume can be found by integrating in cylindrical coordinates: V = □□^{2□} □□^{1} □_{z=r^2}^{4} r dz dr d□.

How do you evaluate the triple integral for this volume explicitly?

First, integrate with respect to z: □_{z=r^2}^{4} dz = 4 - r^2. Then, integrate with respect to r and □: V = □□^{2□} □□^{1} r(4 - r^2) dr d□.

What is the significance of the surfaces $Z = x^2 + y^2$ and $Z = 4$ in defining the shape of the solid?

$Z = x^2 + y^2$ is a paraboloid opening upward, and $Z = 4$ is a horizontal plane; together, they form a cap on the paraboloid within the cylinder, creating a bounded 3D region.

Can the volume of the solid be interpreted as a physical or geometric property?

Yes, it represents the three-dimensional space enclosed between the paraboloid and the plane within the cylinder, which can be interpreted as the volume of a bowl-shaped object capped at $Z = 4$.

What is the result of the volume integral for this solid?

Evaluating the integral: V = □□^{2□} d□ □□^{1} r(4 - r^2) dr = 2□ [2 - (1/2)] = 2□ (3/2) = 3□.

Are there any symmetry considerations that simplify calculating the volume of this solid?

Yes, the symmetry about the z-axis allows us to reduce the problem to a single integral in r and □, simplifying the calculation since the surfaces are rotationally symmetric.

Additional Resources

Consider the solid bounded by surfaces given by $z = x^2 + y^2$, $z = 4$, and within the cylinder $x^2 + y^2 = 1$—this problem offers a rich opportunity to explore the fascinating interplay between surfaces, regions, and volume calculation in multivariable calculus. Understanding how these surfaces interact to define a three-dimensional solid provides insight into geometric intuition, coordinate transformations, and integration techniques.

In this comprehensive guide, we'll dissect this problem step-by-step, covering the geometry of the surfaces involved, setting up the appropriate coordinate systems, defining the bounds precisely, and performing the necessary integrations to compute the volume of the solid. Whether you're a student preparing for an exam or a professional looking to refresh foundational concepts, this detailed analysis aims to clarify each piece of the puzzle.

Understanding the Surfaces and Regions

Before delving into calculations, it's essential to understand the surfaces involved and how they intersect to form the bounded solid.

The Surfaces

1. Paraboloid: $z = x^2 + y^2$
- This surface opens upward, symmetric about the z-axis.
- The lowest point is at the origin (0,0,0).
- For fixed z, the cross-section in the xy-plane is a circle with radius $\sqrt{z}$.

2. Horizontal Plane: $z = 4$
- A flat plane parallel to the xy-plane at height $z = 4$.
- It serves as an upper boundary for the solid.

3. Cylinder: $x^2 + y^2 = 1$

- A circular cylinder of radius 1 centered along the z-axis.
- The boundary in the xy-plane is the circle of radius 1.

The Bounded Region

The solid is bounded by:

- The paraboloid $z = x^2 + y^2$ (below),
- The plane $z = 4$ (above), and
- The cylindrical boundary $x^2 + y^2 = 1$ (in the xy-plane).

This creates a "cap" of the paraboloid truncated at $z = 4$, confined within the cylinder.

Visualizing the Geometry

Imagine a paraboloid rising from the xy-plane, capped at $z=4$. The paraboloid intersects the plane $z=4$ where $x^2 + y^2 = 4$, but since the cylinder's boundary is at radius 1, the intersection within the cylinder occurs only where $x^2 + y^2 \leq 1$.

Within this context:

- The paraboloid $z = x^2 + y^2$ reaches $z=1$ at the boundary $x^2 + y^2=1$, so the paraboloid inside the cylinder extends from $z=0$ to $z=1$ on the boundary.
- The plane $z=4$ is above the paraboloid everywhere within the cylinder, thus forming an upper cap.
- The intersection of the paraboloid and the plane $z=4$ occurs outside the cylinder because at the boundary $x^2 + y^2=1$, $z = 1$, which is below 4.

Thus, the region of the solid is the volume enclosed below the plane $z=4$, above the paraboloid, and within the cylinder $x^2 + y^2$ □ 1.

Formal Mathematical Description of the Region

To set up an integral over this volume, we need to describe it precisely:

- The projection onto the xy-plane is the disk $x^2 + y^2$ □ 1.
- For each point (x, y) in this disk, the z-values range from the paraboloid surface up to the plane:

$$
z_{\text{bottom}} = x^2 + y^2, \quad z_{\text{top}} = 4
$$

- The entire region is therefore:

$$
0 \leq x^2 + y^2 \leq 1, \quad x^2 + y^2 \leq 4, \quad \text{but within the cylinder, so } x^2 + y^2 \leq 1
$$

- Since the paraboloid opens upward and the plane is at z=4, the volume is contained within the cylinder and between these surfaces.

Choosing the Coordinate System: Cylindrical Coordinates

Given the symmetry around the z-axis, cylindrical coordinates (r, □, z) are the most natural choice:

- $x = r \cos \theta$

- $y = r \sin \theta$
- $z = z$

The Jacobian determinant for the transformation is:

$$
dV = r \, dr \, d\theta \, dz
$$

Boundaries in cylindrical coordinates:

- Radial coordinate (r): from 0 to 1 (since the cylinder's radius is 1)
- Angular coordinate (θ): from 0 to 2π
- Vertical coordinate (z): from the paraboloid surface $z = r^2$ up to the plane $z=4$

Setting Up the Volume Integral

The volume V of the solid can be expressed as:

$$
V = \int_{0}^{2\pi} \int_{0}^{1} \int_{z = r^2}^{4} r \, dz \, dr \, d\theta
$$

Note:
- The lower limit for z is the paraboloid $z = r^2$.
- The upper limit for z is the plane $z=4$.
- The radial limit is from 0 to 1, as constrained by the cylinder.
- The angular limit is full rotation from 0 to 2π.

Computing the Volume

Let's proceed step-by-step:

Step 1: Integrate with respect to z

$$
\int_{z = r^2}^{4} r \, dz = r \times (4 - r^2)
$$

Step 2: Integrate with respect to r

$$
\int_{0}^{1} r (4 - r^2) \, dr
$$

Expand the integrand:

$$
\int_{0}^{1} (4r - r^3) \, dr
$$

Compute separately:

$$
\int_{0}^{1} 4r \, dr = 4 \times \frac{r^2}{2} \Big|_{0}^{1} = 4 \times \frac{1}{2} = 2
$$

\[

$$
\int_{0}^{1} r^3 \, dr = \frac{r^4}{4} \Big|_{0}^{1} = \frac{1}{4}
$$

Subtract:

$$
2 - \frac{1}{4} = \frac{8}{4} - \frac{1}{4} = \frac{7}{4}
$$

Step 3: Integrate with respect to □

$$
\int_{0}^{2\pi} d\theta = 2\pi
$$

Final calculation:

$$
V = 2\pi \times \frac{7}{4} = \frac{14\pi}{4} = \frac{7\pi}{2}
$$

Final Result

The volume of the solid bounded by the surfaces $z = x^2 + y^2$, $z=4$, and within the cylinder $x^2 + y^2=1$ is $\boxed{\frac{7\pi}{2}}$.

Additional Considerations and Variations

1. Visualizing Cross-Sections

- Horizontal slices at fixed z:

For $z < 1$, the paraboloid is within the cylinder radius $r = \sqrt{z}$.

For z from 1 to 4, the paraboloid's radius exceeds 1, but since the boundary is the cylinder $r=1$, the paraboloid intersects the cylinder at $r=1$ when $z=1$.

The volume calculation accounts for this, as the integration limits are set accordingly.

2. Alternative Integration Approaches

- Using polar coordinates directly in the xy-plane is equivalent to cylindrical coordinates here due to symmetry.
- For more complex bounds, substitution or change of variables might be necessary.

3. Visual Tools for Better Understanding

- Graphing software can illustrate the paraboloid, cylinder, and plane to visualize the bounded region.
- 3D plots can help verify the bounds and ensure the integral setup captures the entire volume.

Summary of Key Steps

- Identify the surfaces bounding the solid.
- Translate the problem into an appropriate coordinate system (cylindrical coordinates).
- Determine the bounds for each variable based on the geometric intersections.
- Set up the triple integral with the correct limits.
- Perform the integrations step-by-step, simplifying where possible.
- Interpret the result within the context of the geometry.

Final Remarks

Understanding how to analyze and compute the volume of a solid bounded by multiple surfaces is a fundamental skill in multivariable calculus. This problem exemplifies how symmetry, coordinate transformations, and careful boundary analysis come together to yield an elegant solution. By mastering these techniques, you'll be well-equipped to tackle a broad class of volume and surface integral problems with confidence.

Whether you're working

Consider The Solid Bounded By Surfaces Given By Z X2 Y2 Z 4 And Within The Cylinder X2 Y2 1

Consider The Solid Bounded By Surfaces Given By Z X2 Y2 Z 4 And Within The Cylinder X2 Y2 1

Related Articles

- Can The Sticky Ends Created By XhoI And SalI Ligate To Each Other? If Yes, Can The Resulting Sequences
- Describe The Three Kinds Of Plate Boundaries. Explain Your Answer In At Least Three To Four Sentences.
- EXERCISE 4:COMPLETE THE SENTENCES WITH THE WORDS IN BOLD FROM THE STORY.1...........I Woke Up Suddenly

Back to Home