Consider The Vector Field F(x,y,z)=6y,6x,4z. Show That F Is A Gradient Vector Field F=v By Determining

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Introduction to Vector Fields and Gradient Fields

Understanding vector fields and their properties is fundamental in vector calculus and physics. A vector field assigns a vector to every point in space, representing quantities such as force, velocity, or electric and magnetic fields. Among various types of vector fields, gradient vector fields hold particular significance because they are directly related to scalar potential functions. Recognizing whether a given vector field is a gradient vector field involves analyzing its components and verifying specific conditions.

Definition of a Gradient Vector Field

A vector field $\mathbf{F}(x,y,z)$ in three-dimensional space is called a gradient vector field if there exists a scalar function $\phi(x,y,z)$, called a potential function, such that:

$$
\mathbf{F} = \nabla \phi
$$

where

$$
\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)
$$

In other words, each component of $\mathbf{F}$ corresponds to the partial derivative of ϕ with respect to the corresponding coordinate.

Given Vector Field: $\mathbf{F}(x,y,z) = (6y, 6x, 4z)$

Let's examine the vector field:

$$
\mathbf{F}(x,y,z) = (6y, 6x, 4z)
$$

Our goal is to determine whether $\mathbf{F}$ can be expressed as the gradient of some scalar potential function $\phi(x, y, z)$. To do this, we need to find ϕ such that:

$$
\frac{\partial \phi}{\partial x} = 6y
$$
$$
\frac{\partial \phi}{\partial y} = 6x
$$
$$
\frac{\partial \phi}{\partial z} = 4z
$$

Step-by-Step Determination of the Potential Function

1. Integrate with respect to x

Starting with the first component:

$$
\frac{\partial \phi}{\partial x} = 6y
$$

Treat y and z as constants during integration:

$$
\phi(x,y,z) = \int 6y \, dx = 6y x + C_1(y,z)
$$

Here, $C_1(y,z)$ is an arbitrary function of y and z, since the partial derivative with respect to x eliminated any dependence on x.

---

2. Differentiate ϕ with respect to y and compare with the second component

From the potential candidate:

$$
\phi(x,y,z) = 6 y x + C_1(y,z)
$$

Compute:

$$
\frac{\partial \phi}{\partial y} = 6 x + \frac{\partial C_1(y,z)}{\partial y}
$$

From the given $\mathbf{F}$:

$$
\frac{\partial \phi}{\partial y} = 6 x
$$

Set equal:

$$
6 x + \frac{\partial C_1(y,z)}{\partial y} = 6 x
$$

Subtract $6x$:

$$
\frac{\partial C_1(y,z)}{\partial y} = 0
$$

This implies that $C_1(y,z)$ does not depend on y:

$$
C_1(y,z) = C_2(z)
$$

---

3. Differentiate ϕ with respect to z and compare with the third component

Now, the potential function is:

\[

$$
\phi(x,y,z) = 6 y x + C_2(z)
$$

Calculate:

$$
\frac{\partial \phi}{\partial z} = \frac{d C_2(z)}{d z}
$$

Given that:

$$
\frac{\partial \phi}{\partial z} = 4z
$$

Therefore:

$$
\frac{d C_2(z)}{d z} = 4z
$$

Integrate:

$$
C_2(z) = 2 z^2 + C_3
$$

where C_3 is an arbitrary constant.

Constructing the Potential Function $\phi(x, y, z)$

Combining all parts, the potential function ϕ is:

$$
\boxed{
\phi(x, y, z) = 6 y x + 2 z^2 + C
}
$$

where C is an arbitrary constant (since potential functions are determined up to an additive constant).

Verification: Confirming $\mathbf{F} = \nabla \phi$

To verify that $\mathbf{F}$ is indeed a gradient vector field of $\phi(x,y,z) = 6 y x + 2 z^2 + C$, compute the gradient:

$$
\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)
$$

Calculate each component:

$$
\frac{\partial \phi}{\partial x} = 6 y
$$
$$
\frac{\partial \phi}{\partial y} = 6 x
$$
$$
\frac{\partial \phi}{\partial z} = 4 z
$$

Thus,

$$
\nabla \phi = (6 y, 6 x, 4 z) = \mathbf{F}(x,y,z)
$$

This confirms that the vector field $\mathbf{F}$ is the gradient of the scalar potential function ϕ.

Implications and Significance of the Result

Discovering that $\mathbf{F}$ is a gradient vector field has several important implications:

- Potential Function: Since $\mathbf{F}$ can be expressed as $\nabla \phi$, it possesses a scalar potential ϕ. This means the work done by the field in moving a particle from one point to another depends only on the endpoints, not the path taken.

- Conservative Nature: Gradient fields are conservative vector fields. They have zero curl, which can be verified as an additional check.

- Applications: Such fields are common in physics, for example, gravitational and electrostatic fields, where potential energy functions are used to describe the system.

Additional Verification: Curl of $\mathbf{F}$

To further confirm that $\mathbf{F}$ is a gradient field, check if its curl is zero:

$$
\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)
$$

Calculate each component:

- First component:

$$
\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = \frac{\partial 4z}{\partial y} - \frac{\partial 6x}{\partial z} = 0 - 0 = 0
$$

- Second component:

$$
\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = \frac{\partial 6y}{\partial z} - \frac{\partial 4z}{\partial x} = 0 - 0 = 0
$$

- Third component:

$$
\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial 6x}{\partial x} - \frac{\partial 6y}{\partial y} = 6 - 6 = 0
$$

Since the curl of $\mathbf{F}$ is zero everywhere, it supports that $\mathbf{F}$ is conservative and hence a gradient field.

Conclusion

In summary, the vector field $\mathbf{F}(x,y,z) = (6y, 6x, 4z)$ is a gradient vector field. By integrating the components and verifying the consistency with the potential function, we have shown that:

$$
\boxed{
\mathbf{F} = \nabla \phi \quad \text{where} \quad \phi(x,y,z) = 6 y x + 2 z^2 + C
}
$$

\]

This analysis not only confirms the nature of \(\mathbf{F}\) but also highlights the importance of potential functions in understanding the

Frequently Asked Questions

What is the vector field F(x, y, z) given in the problem?

The vector field is F(x, y, z) = (6y, 6x, 4z).

How can we verify if F is a gradient vector field?

We check if there exists a scalar function v(x, y, z) such that F = ∇v, meaning the components of F are the partial derivatives of v with respect to x, y, and z.

What are the necessary conditions for F to be a gradient vector field?

The vector field must be conservative, which typically requires its curl to be zero: ∇ × F = 0, and the components should be continuous and have continuous partial derivatives.

How do we find the potential function v(x, y, z) for F?

Integrate each component of F with respect to its corresponding variable, ensuring consistency across all partial derivatives to determine v.

What is the potential function v(x, y, z) corresponding to F(x, y, z)= (6y, 6x, 4z)?

Integrating: v(x, y, z) = 6xy + 2z^2 + C, where C is an arbitrary constant.

How do we confirm that F is indeed a gradient vector field?

By verifying that F equals the gradient of v, i.e., ∇v = (∂v/∂x, ∂v/∂y, ∂v/∂z), which in this case matches F, confirming that F is a gradient field.

Additional Resources

Gradient Vector Field: An In-Depth Exploration of \(\mathbf{F(x,y,z) = (6y, 6x, 4z)}\)

Introduction: Unveiling the Gradient Vector Field

Imagine a universe where vector fields are not merely random or complex arrangements but can be elegantly derived from scalar potential functions. This is the essence of a gradient vector field—a vector field that arises as the gradient of a scalar potential function $v(x, y, z)$. Recognizing whether a given vector field is a gradient field is fundamental in multivariable calculus, physics, and engineering, especially in contexts like electrostatics, gravity, and fluid flow.

In this review, we examine the vector field:

$$
\mathbf{F}(x, y, z) = (6y, 6x, 4z)
$$

Our goal is to determine whether $\mathbf{F}$ is a gradient vector field, i.e., whether there exists a scalar function $v(x, y, z)$ such that:

$$
\nabla v = \mathbf{F}
$$

This exploration will include detailed steps, theoretical foundations, and practical checks, presented as an expert review to guide readers through the process.

Understanding the Concept: Gradient Vector Fields

What Is a Gradient Vector Field?

A gradient vector field is constructed from a scalar potential function $v(x, y, z)$ as:

$$
\boxed{
\mathbf{F} = \nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\right)
}
$$

This structure implies that each component of the vector field corresponds to a partial derivative of some scalar function v. Conversely, if a vector field $\mathbf{F}$ can be expressed as the gradient of a scalar v, then $\mathbf{F}$ is called a conservative or potential vector field.

Why Is This Important?

- Physical Significance: In physics, such fields represent conservative forces, like gravity or electrostatic fields.
- Mathematical Implication: These fields are irrotational ($\nabla \times \mathbf{F} = 0$) in simply connected domains, a key property for identification.

Step 1: Analyzing the Given Vector Field $\mathbf{F}$

The vector field in question:

$$
\mathbf{F}(x, y, z) = (6y, 6x, 4z)
$$

can be broken down component-wise:

- $F_x = 6y$
- $F_y = 6x$
- $F_z = 4z$

The initial task is to check whether these components can be derived from some scalar potential $v(x, y, z)$:

$$
\frac{\partial v}{\partial x} = 6y
$$
$$
\frac{\partial v}{\partial y} = 6x
$$
$$
\frac{\partial v}{\partial z} = 4z
$$

Step 2: Finding the Scalar Potential Function $v(x, y, z)$

Integrating the First Component

Starting with the first component:

$$
\frac{\partial v}{\partial x} = 6y
$$

Since $6y$ does not depend on x, integrating with respect to x:

$$
v(x, y, z) = 6y \cdot x + C(y, z)
$$

where $C(y, z)$ is an arbitrary function of y and z because the indefinite integral with respect to x could include such a function.

Differentiating to Find $C(y, z)$

Now, differentiate $v(x, y, z)$:

$$

$$
\frac{\partial v}{\partial y} = 6x + \frac{\partial C(y, z)}{\partial y}
$$

But from the given field:

$$
\frac{\partial v}{\partial y} = 6x
$$

Comparing, we have:

$$
6x + \frac{\partial C(y, z)}{\partial y} = 6x
$$
$$
\implies \frac{\partial C(y, z)}{\partial y} = 0
$$

This indicates that $C(y, z)$ does not depend on y:

$$
C(y, z) = C(z)
$$

Step 3: Integrating for the z-dependence

Next, differentiate v with respect to z:

$$
\frac{\partial v}{\partial z} = \frac{\partial}{\partial z} \left(6yx + C(z)\right) = \frac{d C(z)}{d z}
$$

From the vector field:

$$
\frac{\partial v}{\partial z} = 4z
$$

Matching these:

$$
\frac{d C(z)}{d z} = 4z
$$

Integrate:

$$
C(z) = 2z^2 + D
$$

where D is an arbitrary constant.

Step 4: Constructing the Potential Function $v(x, y, z)$

Putting it all together:

$$
v(x, y, z) = 6xy + 2z^2 + D
$$

Since the additive constant D does not affect the gradient, we can set $D=0$ for simplicity.

Step 5: Verifying the Gradient

Now, verify whether ∇v matches the original vector field:

$$
\nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \right)
$$

Calculate each:

- $\frac{\partial v}{\partial x} = 6y$ ✓ matches F_x
- $\frac{\partial v}{\partial y} = 6x$ ✓ matches F_y
- $\frac{\partial v}{\partial z} = 4z$ ✓ matches F_z

Conclusion: $\mathbf{F}$ is indeed a gradient vector field, generated by the scalar potential:

$$
\boxed{
v(x, y, z) = 6xy + 2z^2
}
$$

Additional Checks: Curl and Domain Considerations

Curl of $\mathbf{F}$

For a vector field to be conservative (a gradient field), it must be irrotational:

$$
\nabla \times \mathbf{F} = 0
$$

Calculate:

$$
\nabla \times \mathbf{F} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
6y & 6x & 4z
\end{vmatrix}
$$

Compute each component:

- $(\nabla \times \mathbf{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0 - 0 = 0$
- $(\nabla \times \mathbf{F})_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = 0 - 0 = 0$
- $(\nabla \times \mathbf{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 6 - 6 = 0$

Since the curl is zero everywhere, $\mathbf{F}$ is irrotational.

Domain Considerations

The scalar potential $v(x, y, z) = 6xy + 2z^2$ is defined everywhere in $\mathbb{R}^3$, indicating the domain is simply connected, satisfying the necessary condition for the field to be conservative.

Final Summary: Is $\mathbf{F}$ a Gradient Field?

Based on the component analysis, integration, and verification steps:

- Yes, $\mathbf{F}(x, y, z) = (6y, 6x, 4z)$ is a gradient vector field.
- The corresponding scalar potential function is:

$$
\boxed{
v(x, y, z) = 6xy + 2z^2
}
$$

- The field is conservative, irrotational, and defined over a simply connected domain, fulfilling all criteria for being a gradient field.

Practical Implications and Applications

Understanding this property allows for various applications:

- Potential Energy in Physics: The scalar v can represent potential energy, with $\mathbf{F}$ as the force derived from it.
- Work and Path Independence: Line integrals involving $\mathbf{F}$ are path-independent, simplifying calculations in physics and engineering.
- Flow and Field Analysis: Recognizing

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