Determine The Differential Of Arc Length For The Curve C Parametrized Byr(t) = (e^, Ln(t + 1), 2 - T),

Determine The Differential Of Arc Length For The Curve C Parametrized By r(t) = (e^t, ln(t + 1), 2 - t)

Understanding how to compute the differential of arc length for a parametrized curve is fundamental in calculus, especially in the study of curves in multivariable calculus. This process allows us to measure infinitesimal segments of the curve, which can be integrated to find the total length of the curve over a given interval. In this article, we will perform a detailed analysis of how to determine the differential of arc length for the specific curve \(C \) parametrized by the vector function \(\mathbf{r}(t) = (e^t, \ln(t + 1), 2 - t) \).

Introduction to Arc Length and Its Differential

Before delving into the specific curve, it is essential to understand the general concept of arc length and its differential in the context of parametrized curves.

What Is Arc Length?

Arc length refers to the distance along a curve between two points. For a curve \(C \) parametrized by a vector function \(\mathbf{r}(t) \), where \(t \) varies over an interval \([a, b] \), the total arc length \(s \) from \(t = a \) to \(t = b \) is given by the integral:

\[
s = \int_a^b |\mathbf{r}'(t)| \, dt
\]

where \(|\mathbf{r}'(t)| \) is the magnitude of the velocity vector \(\mathbf{r}'(t) \).

Differential of Arc Length

The differential of arc length, denoted as \(ds \), represents an infinitesimal segment of the curve corresponding to a small change in the parameter t. It is expressed as:

$$
ds = |\mathbf{r}'(t)| \, dt
$$

This differential allows us to approximate the length of very small segments and is fundamental in setting up integrals for total length or in applications like calculating the length of a curve in physics and geometry.

Step-by-Step Calculation for the Given Curve

Given the parametrization:

$$
\mathbf{r}(t) = \left(e^t, \ln(t + 1), 2 - t \right)
$$

we aim to find the differential of arc length ds.

Step 1: Find the Derivative $\mathbf{r}'(t)$

Differentiate each component of $\mathbf{r}(t)$ with respect to t:

$$
\mathbf{r}'(t) = \left(\frac{d}{dt} e^t, \frac{d}{dt} \ln(t + 1), \frac{d}{dt} (2 - t) \right)
$$

Calculating each derivative:

- $\frac{d}{dt} e^t = e^t$
- $\frac{d}{dt} \ln(t + 1) = \frac{1}{t + 1}$
- $\frac{d}{dt} (2 - t) = -1$

Thus,

$$
\mathbf{r}'(t) = \left(e^t, \frac{1}{t + 1}, -1 \right)
$$

Step 2: Compute the Magnitude $|\mathbf{r}'(t)|$

The magnitude of the derivative vector is:

$$
|\mathbf{r}'(t)| = \sqrt{ \left(e^t \right)^2 + \left(\frac{1}{t + 1} \right)^2 + (-1)^2 }
$$

Simplify each term:

$$
|\mathbf{r}'(t)| = \sqrt{ e^{2t} + \frac{1}{(t + 1)^2} + 1 }
$$

This expression represents the rate at which the arc length accumulates at the parameter value t.

Step 3: Write the Differential ds

Using the magnitude, the differential of the arc length is:

$$
ds = |\mathbf{r}'(t)| \, dt = \sqrt{ e^{2t} + \frac{1}{(t + 1)^2} + 1 } \, dt
$$

This is the core formula needed to compute the differential of the arc length for the given curve.

Interpreting and Applying the Differential of Arc Length

Understanding the differential ds allows for multiple applications, including:

- Computing the total length of the curve over an interval.
- Analyzing how the length varies with respect to the parameter.
- Facilitating calculations in physics, such as work or energy along a path.

Calculating the Total Arc Length

The total length S of the curve from $t = t_0$ to $t = t_1$ is obtained by integrating ds:

$$
S = \int_{t_0}^{t_1} |\mathbf{r}'(t)| \, dt = \int_{t_0}^{t_1} \sqrt{ e^{2t} + \frac{1}{(t + 1)^2} + 1 } \, dt
$$

Depending on the interval, this integral may need to be evaluated numerically due to the complexity of the integrand.

Special Considerations and Domain of the Parameter t

When working with the given parametrization, it is crucial to consider the domain of t:

- The term $\ln(t + 1)$ requires $t + 1 > 0 \Rightarrow t > -1$.
- The derivatives are well-defined for $t > -1$.

Therefore, the natural domain for the parameter t is:

$$
t \in (-1, \infty)
$$

Within this interval, the differential ds is valid and can be used to compute arc lengths or analyze the curve's behavior.

Practical Examples and Numerical Computation

Suppose we want to find the length of the curve between $t = 0$ and $t = 2$:

$$
S = \int_0^2 \sqrt{ e^{2t} + \frac{1}{(t + 1)^2} + 1 } \, dt
$$

This integral does not have an elementary antiderivative, so numerical methods such as Simpson's rule or numerical integration software (e.g., MATLAB, WolframAlpha, or Python's SciPy library) are employed.

Example: Numerical approximation of the arc length

Using a computational tool, you can estimate:

```python
import scipy.integrate as integrate
import numpy as np

def integrand(t):
return np.sqrt(np.exp(2t) + 1/((t + 1)2) + 1)

length, error = integrate.quad(integrand, 0, 2)
print(f"Approximate arc length: {length}")
```

This approach provides a practical method for handling complex integrals in real-world applications.

Summary and Key Takeaways

- The differential of arc length ds for a parametrized curve $\mathbf{r}(t)$ is given by:

$$
ds = |\mathbf{r}'(t)| \, dt
$$

- For $\mathbf{r}(t) = (e^t, \ln(t + 1), 2 - t)$, the derivative vector is:

$$
\mathbf{r}'(t) = \left(e^t, \frac{1}{t + 1}, -1 \right)
$$

- The magnitude of this derivative is:

$$
|\mathbf{r}'(t)| = \sqrt{ e^{2t} + \frac{1}{(t + 1)^2} + 1 }
$$

- The differential of arc length thus becomes:

$$
ds = \sqrt{ e^{2t} + \frac{1}{(t + 1)^2} + 1 } \, dt
$$

- To find the total length over an interval, integrate $|\mathbf{r}'(t)|$ over that interval.

Conclusion

Determining the differential of arc length for a given parametrized curve is a fundamental skill in calculus, enabling the calculation of lengths, analysis of curves, and application in physics and engineering problems. By carefully differentiating each component, computing the magnitude of the derivative vector, and integrating it over the desired interval, we can accurately measure the length of complex curves such as $\mathbf{r}(t) = (e^t, \ln(t + 1), 2 - t)$. Mastery of these techniques provides a strong foundation for advanced studies in multivariable calculus and related fields.

Frequently Asked Questions

What is the general formula for the differential of arc length for a parametrized curve?

The differential of arc length ds for a curve parametrized by $r(t) = (x(t), y(t), z(t))$ is given by $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}\, dt$.

How do you find the derivatives of each component for the curve $r(t) = (e^t, \ln(t + 1), 2 - t)$?

The derivatives are: $dx/dt = e^t$, $dy/dt = 1/(t + 1)$, $dz/dt = -1$.

What is the expression for the differential of arc length for the given curve $r(t) = (e^t, \ln(t + 1), 2 - t)$?

$ds = \sqrt{(e^t)^2 + (1/(t + 1))^2 + (-1)^2}\, dt = \sqrt{e^{2t} + 1/(t + 1)^2 + 1}\, dt$.

Over what interval should t be considered to compute the arc length for this curve?

t should be considered over the interval where the curve is defined, typically $t > -1$ for $\ln(t + 1)$, depending on the specific segment of the curve being analyzed.

How does the exponential component e^t affect the differential of arc length?

Since e^t appears squared in the expression (e^{2t}), it exponentially influences the magnitude of the differential ds, increasing rapidly as t increases.

Why is it important to include the derivative of each component when determining ds?

Including each derivative ensures an accurate calculation of the rate of change of the curve in all directions, which is essential for computing the correct differential of arc length.

Can the differential of arc length be used to find the total length of the curve between two points?

Yes, by integrating ds from the starting to ending parameter values, you can find the total length of the curve segment.

What challenges might arise when computing the integral for the arc length of this curve?

The integral involves a square root of a sum of functions, including e^{2t} and 1/(t+1)^2, which may require special techniques or numerical methods to evaluate.

How does the parameter t influence the shape of the curve r(t) = (e^t, ln(t + 1), 2 - t)?

The parameter t affects the exponential growth in x, the logarithmic variation in y, and linear decrease in z, shaping the 3D trajectory of the curve.

What steps are necessary to compute the differential of arc length for the given curve?

First, find the derivatives of each component with respect to t, then substitute into the ds formula to get the integrand, and finally evaluate the integral over the desired interval.

Additional Resources

Determine The Differential Of Arc Length For The Curve C Parametrized By r(t) = (e^t, ln(t + 1), 2 - t)

Introduction

In the realm of calculus and differential geometry, understanding how to compute the differential of arc length for a given curve is fundamental. This process not only deepens our comprehension of the geometric properties of curves but also has practical applications in physics, engineering, and computer graphics. Today, we will explore this concept through a detailed analysis of the curve C parametrized by the vector function:

$$
\mathbf{r}(t) = (e^t, \ln(t + 1), 2 - t)
$$

Our goal is to determine the differential of the arc length along this curve, which involves calculating the derivative of the position vector with respect to the parameter t, and then formulating the differential arc length expression. This process illuminates how infinitesimal changes in the parameter t relate to the actual length along the curve, offering insight into the curve’s geometry.

Understanding the Concept of Arc Length and Its Differential

What Is Arc Length?

Arc length measures the distance along a curve between two points. If a curve is described parametrically, the total arc length s from a point $t = a$ to $t = b$ is given by the integral:

$$
s = \int_{a}^{b} |\mathbf{r}'(t)| \, dt
$$

where $\mathbf{r}'(t)$ denotes the derivative of the position vector with respect to t. The magnitude $|\mathbf{r}'(t)|$ essentially captures how fast the curve is traversed at each point.

Differential of Arc Length

The differential of arc length, ds, represents an infinitesimal stretch of the curve corresponding to an infinitesimal change dt in the parameter. Formally, it can be expressed as:

$$
ds = |\mathbf{r}'(t)| \, dt
$$

This relation is pivotal because it allows us to convert infinitesimal changes in the parameter t into actual length increments along the curve.

Computing the Derivative of the Curve $\mathbf{r}(t)$

Step 1: Derive each component

Given:

$$
\mathbf{r}(t) = (e^t, \ln(t + 1), 2 - t)
$$

we differentiate each component with respect to t:

- For the first component:

$$
\frac{d}{dt} e^t = e^t
$$

- For the second component:

$$
\frac{d}{dt} \ln(t + 1) = \frac{1}{t + 1}
$$

- For the third component:

$$
\frac{d}{dt} (2 - t) = -1
$$

Step 2: Form the derivative vector $\mathbf{r}'(t)$

Putting these together, the derivative of the vector function is:

$$
\mathbf{r}'(t) = \left(e^t, \frac{1}{t + 1}, -1 \right)
$$

This derivative encapsulates the instantaneous rate of change of the position vector along the curve.

Calculating the Magnitude of $\mathbf{r}'(t)$

The next step involves calculating the magnitude $|\mathbf{r}'(t)|$, which is crucial for determining the differential arc length.

$$
|\mathbf{r}'(t)| = \sqrt{\left(e^t\right)^2 + \left(\frac{1}{t + 1}\right)^2 + (-1)^2}
$$

Simplify:

$$
|\mathbf{r}'(t)| = \sqrt{ e^{2t} + \frac{1}{(t + 1)^2} + 1 }
$$

This expression provides the rate at which the curve is traversed at any point t.

Formulating the Differential of Arc Length ds

Having obtained $|\mathbf{r}'(t)|$, the differential of arc length along the curve becomes:

$$
ds = |\mathbf{r}'(t)| \, dt = \sqrt{ e^{2t} + \frac{1}{(t + 1)^2} + 1 } \, dt
$$

This formula allows us to compute an infinitesimal segment of the curve’s length for a small change in t.

Interpreting and Applying the Result

Physical and Geometric Significance

The expression for ds encapsulates how the curve stretches and compresses in space as t varies. For example, when t is large, e^{2t} dominates, indicating rapid growth in the first component, influencing the overall length increment. Conversely, near $t = -1$, the term $\frac{1}{(t + 1)^2}$ becomes significant, highlighting the importance of the logarithmic component.

Practical Applications

- Calculating Total Arc Length: To find the total length of the curve between two points $t = a$ and $t = b$, integrate:

$$
s = \int_{a}^{b} \sqrt{ e^{2t} + \frac{1}{(t + 1)^2} + 1 } \, dt
$$

- Design and Engineering: Engineers use such calculations to determine distances along complex paths, such as roads, pipelines, or robotic arms.

- Physics and Motion: Understanding how an object moves along a path involves analyzing the differential of arc length to relate time and distance traveled.

Note on Integration

While the differential ds provides the infinitesimal length element, calculating total length involves integration, which may require numerical methods depending on the limits and the integrand's complexity.

Special Considerations and Constraints

Domain of t

Since the second component involves $\ln(t + 1)$, the domain of the parameter t must satisfy:

$$
t + 1 > 0 \quad \Rightarrow \quad t > -1
$$

This restriction ensures the logarithm is defined and the derivative components are valid.

Behavior at the Domain Boundary

As $t \to -1^+$, $\ln(t + 1) \to -\infty$, and the derivative's second component becomes unbounded. Careful analysis is required when approaching this boundary to understand the curve's behavior.

Conclusion: The Essence of the Calculation

Determining the differential of arc length for the curve $\mathbf{r}(t) = (e^t, \ln(t + 1), 2 - t)$ involves a systematic process:

1. Deriving each component of the vector function.
2. Calculating the magnitude of the derivative vector.
3. Expressing the differential of arc length as $ds = |\mathbf{r}'(t)| \, dt$.

This process bridges the abstract mathematical representation of a curve with tangible geometric intuition, enabling precise measurements of length and facilitating various practical applications. As calculus tools continue to evolve, such fundamental techniques remain central to exploring the fascinating geometry of space curves, guiding engineers, physicists, and mathematicians alike in their pursuit of understanding the world in dimensions both theoretical and applied.

Final Note

Understanding the differential of arc length is more than an academic exercise; it’s a gateway to mastering the geometry of curves, analyzing motion, and designing complex systems. Whether assessing the length of a winding road or modeling particle trajectories, the core principles outlined here serve as the foundation for countless scientific and engineering endeavors.

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