Evaluate The Line Integral, Where Is The Given Curve. Z Dx + X2 Dy + Y2 Dz, C Is The Line Segment From

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Understanding line integrals is fundamental in vector calculus, especially when analyzing the work done by a force field along a specific path. In this comprehensive guide, we will explore how to evaluate the line integral of the vector field $\mathbf{F} = Z \, \mathbf{dx} + X^2 \, \mathbf{dy} + Y^2 \, \mathbf{dz}$, where C is a line segment from a specified starting point to an endpoint. Our focus will be on breaking down the process systematically, illustrating with examples, and providing tips for accurate computation.

Understanding Line Integrals and Their Significance

What Is a Line Integral?

A line integral measures the accumulation of a quantity along a curve in space. It extends the concept of a definite integral from a one-dimensional domain to a curve in two or three dimensions. Specifically, for a vector field $\mathbf{F} = P \, \mathbf{i} + Q \, \mathbf{j} + R \, \mathbf{k}$, the line integral over a curve C parametrized by $\mathbf{r}(t)$ from $t = a$ to $t = b$ is:

$$
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
$$

This integral often represents work done by a force field, flow of a fluid, or other physical quantities along a path.

Applications of Line Integrals

- Calculating work done by a force along a path
- Evaluating circulation of a vector field
- Computing flux across a curve
- Analyzing potential fields in physics and engineering

Given Data and Problem Setup

Suppose you are given a vector field:

$$
\mathbf{F} = Z \, \mathbf{dx} + X^2 \, \mathbf{dy} + Y^2 \, \mathbf{dz}
$$

which can be written in component form as:

$$
\mathbf{F}(x, y, z) = (P, Q, R) = (Z, X^2, Y^2)
$$

The curve C is specified as the line segment from point A to point B:

$$
A = (x_1, y_1, z_1) \quad \text{and} \quad B = (x_2, y_2, z_2)
$$

Our goal is to evaluate:

$$
\int_C \mathbf{F} \cdot d\mathbf{r}
$$

Step-by-Step Process to Evaluate the Line Integral

1. Define the Parametric Equation of the Curve C

To evaluate the integral, first express the line segment as a parametric curve:

$$
\mathbf{r}(t) = (x(t), y(t), z(t))
$$

where $t \in [a, b]$. For a straight line segment from $A = (x_1, y_1, z_1)$ to $B = (x_2, y_2, z_2)$:

$$
x(t) = x_1 + (x_2 - x_1) t
$$

$$
y(t) = y_1 + (y_2 - y_1) t
$$

$$
z(t) = z_1 + (z_2 - z_1) t
$$

with $t \in [0, 1]$.

Tip: Using $t \in [0, 1]$ simplifies calculations, but you can choose any interval that suits your problem.

---

2. Compute $d\mathbf{r}$ and $\mathbf{F}(\mathbf{r}(t))$

Calculate the derivatives:

$$
x'(t) = x_2 - x_1, \quad y'(t) = y_2 - y_1, \quad z'(t) = z_2 - z_1
$$

Evaluate the vector field along $\mathbf{r}(t)$:

$$
\mathbf{F}(\mathbf{r}(t)) = (z(t), x(t)^2, y(t)^2)
$$

---

3. Set Up the Integral

Express the line integral as:

$$
\int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
$$

which becomes:

$$
\int_0^1 \left[Z(t) \, x'(t) + X(t)^2 \, y'(t) + Y(t)^2 \, z'(t) \right] dt
$$

Plug in the parametric expressions:

$$
\int_0^1 \left[z(t) \cdot x'(t) + (x(t))^2 \cdot y'(t) + (y(t))^2 \cdot z'(t) \right] dt

\]

---

4. Perform the Integration

Carry out the integration term by term:

- Integrate \(z(t) \cdot x'(t) \) over \(t \)
- Integrate \((x(t))^2 \cdot y'(t) \) over \(t \)
- Integrate \((y(t))^2 \cdot z'(t) \) over \(t \)

Use substitution where necessary and apply standard integral calculus techniques.

Worked Example: Evaluating the Line Integral

Suppose we have specific points:

\[
A = (1, 0, 2), \quad B = (4, 3, 5)
\]

and the vector field:

\[
\mathbf{F} = Z \, \mathbf{dx} + X^2 \, \mathbf{dy} + Y^2 \, \mathbf{dz}
\]

Step 1: Parametrize the curve \(C \)

\[
x(t) = 1 + 3t, \quad y(t) = 0 + 3t, \quad z(t) = 2 + 3t
\]
where \(t \in [0, 1] \).

Step 2: Compute derivatives

\[
x'(t) = 3, \quad y'(t) = 3, \quad z'(t) = 3
\]

Step 3: Evaluate \(\mathbf{F} \) along the curve

\[
Z = z(t) = 2 + 3t
\]

$$
X^2 = (x(t))^2 = (1 + 3t)^2
$$

$$
Y^2 = (y(t))^2 = (3t)^2 = 9t^2
$$

Step 4: Set up the integral

$$
\int_0^1 \left[z(t) \cdot x'(t) + (x(t))^2 \cdot y'(t) + (y(t))^2 \cdot z'(t) \right] dt
$$

which simplifies to:

$$
\int_0^1 \left[(2 + 3t) \times 3 + (1 + 3t)^2 \times 3 + 9 t^2 \times 3 \right] dt
$$

or

$$
3 \int_0^1 \left[2 + 3t + (1 + 3t)^2 + 9 t^2 \right] dt
$$

Step 5: Expand and integrate

Expand $(1 + 3t)^2 = 1 + 6t + 9 t^2$:

$$
3 \int_0^1 \left[2 + 3t + 1 + 6t + 9 t^2 + 9 t^2 \right] dt
$$

Combine like terms:

$$
3 \int_0^1 \left[(2 + 1) + (3t + 6t) + (9 t^2 + 9 t^2) \right] dt = 3 \int_0^1 \left[3 + 9 t + 18 t^2 \right] dt
$$

Now, integrate term by term:

$$
3 \left[3 t + \frac{9}{2} t^2 + 6 t^3 \right]_0^1
$$

Evaluate at $t=1$:

$$
3 \left[3 \times 1 + \frac{9}{2} \times 1 + 6 \times 1 \right] = 3 \left[3 + 4.5 + 6 \right] = 3 \times 13.5 = 40.
$$

Frequently Asked Questions

How do you evaluate the line integral of the vector field Z dx + X^2 dy + Y^2 dz along a line segment C?

To evaluate the line integral, parametrize the line segment C with a parameter t, express X, Y, Z in terms of t, compute dx, dy, dz, then substitute into the integral and integrate over the parameter's limits.

What is the significance of the curve C in calculating the line integral of the given vector field?

The curve C defines the path along which the line integral is evaluated. Its parametrization determines how the variables X, Y, Z change, affecting the integral's value.

How do I determine the limits of integration for the line segment C in the line integral Z dx + X^2 dy + Y^2 dz?

The limits are determined by the parameter t that describes the endpoints of the line segment. For example, if the segment starts at point A and ends at point B, t typically varies from 0 to 1, with the points expressed as functions of t.

Can you explain the process of choosing a suitable parametrization for the line segment in this integral?

Yes, choose a parametrization that linearly interpolates between the endpoints of the segment. For points (x_1, y_1, z_1) and (x_2, y_2, z_2), set $X = x_1 + (x_2 - x_1)t$, $Y = y_1 + (y_2 - y_1)t$, $Z = z_1 + (z_2 - z_1)t$, with t in [0,1].

What are common mistakes to avoid when evaluating line integrals of this form?

Common mistakes include incorrect parametrization, forgetting to differentiate the parametrized variables (dx, dy, dz), misapplying the limits of integration, or substituting the variables incorrectly into the integral.

Additional Resources

Evaluate The Line Integral, Where Is The Given Curve. Z Dx + X^2 Dy + Y^2 Dz, C Is The Line Segment From — this is a common problem encountered in multivariable calculus, particularly when dealing with vector calculus and line integrals. Understanding how to evaluate such integrals is essential for applications in physics, engineering, and mathematics, such as calculating work done by a force field or flux through a curve. In this article, we will provide a comprehensive guide to understanding, setting up, and computing the line integral of the form:

$$
\int_{C} Z \, dx + X^2 \, dy + Y^2 \, dz
$$

where C is a specified curve, in this case, a line segment from one point to another.

Understanding the Line Integral and Its Components

What Is a Line Integral?

A line integral is a type of integral that involves integrating a function along a curve C. Unlike definite integrals over an interval, line integrals extend the concept to functions defined on a path in space. They often represent physical quantities such as work done by a force field, mass of a wire, or flux of a vector field.

The general form of a line integral of a vector field $\mathbf{F} = P \, \mathbf{i} + Q \, \mathbf{j} + R \, \mathbf{k}$ along a curve C parametrized by $\mathbf{r}(t) = (x(t), y(t), z(t))$, with t in $[a, b]$, is:

$$
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{a}^{b} \left[P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right] dt
$$

In our case, the integral $Z \, dx + X^2 \, dy + Y^2 \, dz$ can be viewed as a line integral of a vector field:

$$
\mathbf{F}(x, y, z) = (Z, X^2, Y^2)
$$

over the curve C.

The Significance of the Curve C

The curve C determines the limits and the form of the integral. For a line segment, the curve is a straight path between two points, which simplifies the parametrization process.

Setting Up the Problem

Suppose the curve C is a line segment from point $\mathbf{A} = (x_1, y_1, z_1)$ to point $\mathbf{B} = (x_2, y_2, z_2)$. Our goal is to evaluate:

$$
\int_{C} Z \, dx + X^2 \, dy + Y^2 \, dz

\]

Step 1: Parametrize the Curve

To evaluate the line integral, we need to express x, y, z as functions of a parameter t, where t varies from a to b.

For a line segment from $\mathbf{A}$ to $\mathbf{B}$, the standard parametrization is:

$$
\mathbf{r}(t) = \mathbf{A} + t(\mathbf{B} - \mathbf{A}) = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1), z_1 + t(z_2 - z_1))
$$

where:

- $t \in [0, 1]$

- When $t = 0$, $\mathbf{r}(0) = \mathbf{A}$

- When $t = 1$, $\mathbf{r}(1) = \mathbf{B}$

Thus, the parametrization is:

$$
x(t) = x_1 + t (x_2 - x_1) \\
y(t) = y_1 + t (y_2 - y_1) \\
z(t) = z_1 + t (z_2 - z_1)
$$

Step 2: Compute Derivatives

Calculate dx/dt, dy/dt, and dz/dt:

$$
\frac{dx}{dt} = x_2 - x_1 \\
\frac{dy}{dt} = y_2 - y_1 \\
\frac{dz}{dt} = z_2 - z_1
$$

Step 3: Express the Integrand in Terms of t

Substitute $x(t)$, $y(t)$, $z(t)$ into the integrand:

$$
\text{Integrand} = Z \, dx + X^2 \, dy + Y^2 \, dz
$$

which becomes:

$$
Z(t) \frac{dx}{dt} dt + X(t)^2 \frac{dy}{dt} dt + Y(t)^2 \frac{dz}{dt} dt
$$

Note:

- $Z(t) = z(t)$

- $X(t) = x(t)$

- $Y(t) = y(t)$

Step-by-Step Evaluation Process

1. Write down the explicit forms of $x(t), y(t), z(t)$

Given the initial and terminal points, plug in the coordinates.

2. Compute the derivatives $dx/dt, dy/dt, dz/dt$

These are constants for a line segment, simplifying the calculation.

3. Substitute into the integrand

Express $Z(t), X(t)^2, Y(t)^2$ explicitly in terms of t.

4. Integrate over t from 0 to 1

Evaluate:

$$
\int_{0}^{1} \left[Z(t) \frac{dx}{dt} + X(t)^2 \frac{dy}{dt} + Y(t)^2 \frac{dz}{dt} \right] dt
$$

Practical Example

Suppose the line segment C runs from $\mathbf{A} = (1, 2, 3)$ to $\mathbf{B} = (4, 5, 6)$.

Step 1: Parametrize

$$
x(t) = 1 + 3t \\
y(t) = 2 + 3t \\
z(t) = 3 + 3t
$$

with $t \in [0, 1]$.

Step 2: Derivatives

$$
\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 3, \quad \frac{dz}{dt} = 3
$$

Step 3: Express integrand

$$
Z(t) = z(t) = 3 + 3t \\
X(t) = x(t) = 1 + 3t \Rightarrow X(t)^2 = (1 + 3t)^2 \\
Y(t) = y(t) = 2 + 3t \Rightarrow Y(t)^2 = (2 + 3t)^2
$$

The integrand becomes:

$$
(3 + 3t) \times 3 + (1 + 3t)^2 \times 3 + (2 + 3t)^2 \times 3
$$

which simplifies to:

$$
3(3 + 3t) + 3(1 + 3t)^2 + 3(2 + 3t)^2
$$

Computing the Integral

Step 1: Simplify the integrand

$$
3(3 + 3t) = 9 + 9t
$$

$$
3(1 + 3t)^2 = 3(1 + 6t + 9t^2) = 3 + 18t + 27 t^2
$$

$$
3(2 + 3t)^2 = 3(4 + 12t + 9t^2) = 12 + 36t + 27 t^2
$$

Adding all:

$$
(9 + 9t) + (3 + 18t + 27 t^2) + (12 + 36t + 27 t^2) = (9 + 3 + 12) + (9t + 18t + 36t) + (27 t^2 + 27 t^2)
$$

\]

\[
= 24 + 63 t + 54 t^2
\]

Step 2: Integrate over \(t \in [0,1] \)

\[
\int_{0}^{1} [24 + 63 t + 54 t^2] dt
\]

Compute:

\[
\int 24 dt = 24 t \\
\int 63 t dt = 31.5 t^2 \\
\int 54 t^2 dt = 18 t^3
\]

Evaluate from 0 to 1:

\[
[24 t +

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