Evaluate The Surface Integral S Sszds, Where S Is The Hemisphere Given By X2 + Y2 + X2 = 1 With Z <

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Introduction to Surface Integrals in Multivariable Calculus

Surface integrals are fundamental tools in multivariable calculus, allowing us to compute quantities such as flux, surface area, and the total amount of a field passing through a surface. They extend the concept of integrals over curves (line integrals) to two-dimensional surfaces embedded in three-dimensional space. Specifically, surface integrals of the form $\iint_S \mathbf{F} \cdot d\mathbf{S}$ or $\iint_S f\, dS$ are used to evaluate the flux of vector fields or scalar fields across surfaces.

In this context, we focus on scalar surface integrals, particularly the integral of a scalar function $f(x, y, z)$ over a surface S. This allows us to determine the total "amount" of the scalar quantity distributed over the surface, which is essential in physics, engineering, and mathematics applications.

Understanding the Surface S: The Hemisphere $X^2 + Y^2 + Z^2 = 1$ with $Z <$

Defining the Surface S

The problem involves a hemisphere defined by the equation:

$$
X^2 + Y^2 + Z^2 = 1
$$

with the condition:

$$
Z < \text{(some value)}
$$

Although the original problem statement appears incomplete, it is standard to interpret such a surface as either the upper or lower hemisphere depending on the context. Typically, the hemisphere is specified as:

- The upper hemisphere: $Z \geq 0$
- The lower hemisphere: $Z \leq 0$

Given the expression "where $Z <$", it is reasonable to assume the problem refers to the lower hemisphere where $Z \leq 0$. For the purpose of this article, we will proceed with the assumption that the surface S is the lower hemisphere of the sphere $x^2 + y^2 + z^2 = 1$.

Formulating the Surface Integral $\iint_S s\, ds$

The integral in question appears as:

$$
\iint_S s\, ds
$$

which suggests integrating the scalar function s over the surface S. Given the notation and typical conventions, it's likely that s is a scalar function, possibly $s = z$ or some other scalar field defined over S.

Assumption: Let's assume the scalar function being integrated is $f(x, y, z) = z$, which is common in such problems. Therefore, the integral becomes:

$$
\iint_S z\, dS
$$

This integral computes the surface integral of the scalar function z over the hemispherical surface S.

Step-by-Step Approach to Evaluating the Surface Integral

To evaluate the integral $\iint_S z\, dS$ over the lower hemisphere, we'll follow these steps:

1. Parameterize the Surface S

2. Express the Surface Element dS in Terms of Parameters

3. Set Up the Integral in the Parameter Domain

4. Compute the Integral

---

1. Parameterization of the Hemisphere

A common way to parameterize a sphere or hemisphere is via spherical coordinates:

$$
\begin{cases}
x = r \sin \phi \cos \theta \\
y = r \sin \phi \sin \theta \\
z = r \cos \phi
\end{cases}
$$

where:

- r is the radius (here $r=1$)
- ϕ is the polar angle (measured from the positive z-axis)
- θ is the azimuthal angle (measured from the positive x-axis in the xy-plane)

For the lower hemisphere of the unit sphere, the parameter domain is:

$$
\begin{cases}
r = 1 \\
\phi \in [\pi/2, \pi] \quad \text{(since \(z \leq 0\))} \\
\theta \in [0, 2\pi]
\end{cases}
$$

Thus, the parametrization simplifies to:

$$
\begin{cases}
x = \sin \phi \cos \theta \\
y = \sin \phi \sin \theta \\
z = \cos \phi
\end{cases}
$$

with $\phi \in [\pi/2, \pi]$ and $\theta \in [0, 2\pi]$.

---

2. Surface Element dS

For a sphere parameterized as above, the surface element is:

$$
dS = r^2 \sin \phi\, d\phi\, d\theta
$$

Since $r=1$, it simplifies to:

$$
dS = \sin \phi\, d\phi\, d\theta

\]

The integrand \(z = \cos \phi\), so the integral becomes:

\[
\iint_S z\, dS = \int_{0}^{2\pi} \int_{\pi/2}^{\pi} \cos \phi \, \sin \phi\, d\phi\, d\theta
\]

---

3. Setting Up the Integral

The integral over the lower hemisphere:

\[
\boxed{
\iint_S z\, dS = \int_{0}^{2\pi} \int_{\pi/2}^{\pi} \cos \phi\, \sin \phi\, d\phi\, d\theta
}
\]

Note that the \(\theta\)-integral is straightforward:

\[
\int_{0}^{2\pi} d\theta = 2\pi
\]

and the remaining integral over \(\phi\):

\[
I = \int_{\pi/2}^{\pi} \cos \phi\, \sin \phi\, d\phi
\]

---

4. Computing the Integral

The integral \(I\) can be evaluated using substitution:

Let:

\[
u = \sin \phi \quad \Rightarrow \quad du = \cos \phi\, d\phi
\]

When \(\phi = \pi/2\):

\[
u = \sin (\pi/2) = 1
\]

When \(\phi = \pi\):

\[

$$
u = \sin \pi = 0
$$

Therefore,

$$
I = \int_{u=1}^{0} u\, du = - \int_{0}^{1} u\, du = - \left[\frac{u^2}{2} \right]_0^1 = - \frac{1}{2}
$$

Putting it all together:

$$
\iint_S z\, dS = 2\pi \times \left(- \frac{1}{2} \right) = -\pi
$$

Final Result and Interpretation

The evaluated surface integral of z over the lower hemisphere of the unit sphere is:

$$
\boxed{
\iint_S z\, dS = -\pi
}
$$

This negative value indicates that, on average, the z-component over the surface is negative, consistent with integrating over the lower hemisphere where $z \leq 0$.

Applications of Surface Integrals in Physics and Engineering

Surface integrals like the one evaluated here are extensively used in various fields:

- Electromagnetism: Calculating flux of electric or magnetic fields across surfaces.
- Fluid Dynamics: Computing flow rates across surfaces.
- Surface Area Calculations: Determining the surface area of complex shapes.
- Heat Transfer: Analyzing heat flux through surfaces.

Understanding how to evaluate such integrals over curved surfaces like hemispheres enables scientists and engineers to model real-world phenomena accurately.

Summary of Key Steps for Evaluating Surface Integrals Over Hemispheres

- Recognize the geometric shape and the relevant domain for parameters.
- Choose an appropriate parameterization (spherical coordinates are standard for spheres).
- Express the surface element dS in terms of parameters.
- Set up the integral with correct bounds.
- Simplify and evaluate the integral, often using substitution.

This systematic approach simplifies complex surface integrals and provides clear insight into the physical or mathematical quantity being calculated.

Conclusion

In conclusion, evaluating the surface integral $\iint_S z\, dS$ over the lower hemisphere of the unit sphere involves parameterizing the surface, expressing the integrand and surface element in terms of spherical coordinates, and performing the integral over the specified domain. The result, $-\pi$, highlights the significance of orientation and the geometric nature of the surface, which are crucial considerations in the application of surface integrals across scientific disciplines.

Keywords: surface integral, hemisphere, spherical coordinates, flux,

Frequently Asked Questions

What is the correct surface integral to evaluate over the hemisphere S given by $x^2 + y^2 + z^2 = 1$ with $z < 0$?

The surface integral is $\iint_S z \, ds$, where S is the lower hemisphere of the unit sphere defined by $x^2 + y^2 + z^2 = 1$ with $z < 0$.

How do you parameterize the hemisphere $x^2 + y^2 + z^2 = 1$ with $z < 0$ for evaluating the surface integral?

Use spherical coordinates: $x = \sin\theta \cos\varphi$, $y = \sin\theta \sin\varphi$, $z = -\cos\theta$, with θ from 0 to $\pi/2$ and φ from 0 to 2π, to parameterize the lower hemisphere.

What is the expression for the surface element ds on the hemisphere in spherical coordinates?

The surface element ds on the sphere is $\sin\theta \, d\theta \, d\varphi$, multiplied by the radius squared (which is 1), so $ds = \sin\theta \, d\theta \, d\varphi$ for the unit sphere.

How do you set up the surface integral $\iint_S z \, ds$ over the given hemisphere?

Express z as $-\cos\theta$, and ds as $\sin\theta \, d\theta \, d\varphi$, then integrate over θ from 0 to π/2 and φ from 0 to 2π: $\iint_S z \, ds = \int_0^{2\pi} \int_0^{\pi/2} (-\cos\theta) \sin\theta \, d\theta \, d\varphi$.

What is the value of the integral $\int_0^{\pi/2} -\cos\theta \sin\theta \, d\theta$?

Evaluating the integral: $\int_0^{\pi/2} -\cos\theta \sin\theta \, d\theta = -\frac{1}{2}$, since $\int \cos\theta \sin\theta \, d\theta = \frac{1}{2} \sin^2\theta$, and at the limits, it evaluates to ½.

What is the final value of the surface integral $\iint_S z \, ds$ over the lower hemisphere?

The integral over φ from 0 to 2π contributes a factor of 2π, so the total integral is $2\pi \times (-\frac{1}{2}) = -\pi$.

Why does the surface integral $\iint_S z \, ds$ over the lower hemisphere result in $-\pi$?

Because $z = -\cos\theta$ contributes negatively over the lower hemisphere, and the integral simplifies to −½ times the integral over φ, which yields −π.

What is the significance of the negative sign in the result of the surface integral over the hemisphere?

The negative sign indicates that the average value of z over the lower hemisphere is negative, consistent with it being below the xy-plane.

Additional Resources

Surface Integral Evaluation

Introduction to Surface Integrals and Their Significance

Surface integrals are fundamental concepts in multivariable calculus, serving as powerful tools to evaluate quantities that are distributed over a surface. Unlike traditional integrals, which integrate over an interval or a region in the plane, surface integrals extend this idea into three-dimensional space, allowing us to compute flux, surface area, and other physical quantities across surfaces.

Understanding how to evaluate a surface integral such as $\iint_{S} z \, ds$ requires not only mastery of calculus but also a keen insight into the geometric properties of the surface in question. In this article, we delve into the detailed process of evaluating the surface integral \(\iint_{S} z

$\, ds$), where S is the upper hemisphere described by $x^2 + y^2 + z^2 = 1$ with $z \geq 0$.

Describing the Surface S: The Hemisphere

Geometric Description of S

The surface S is specified as:

$$
x^2 + y^2 + z^2 = 1, \quad z \geq 0
$$

This is the upper hemisphere of a unit sphere centered at the origin. It represents all points in 3D space that are exactly one unit away from the origin, with z positive or zero.

Key features:

- Radius: 1
- Center: at the origin $(0, 0, 0)$
- Domain in z: from 0 to 1
- Projection onto the xy-plane: the unit disk $x^2 + y^2 \leq 1$

Parameterization of S

To evaluate the surface integral, a common approach is to parameterize S. Spherical coordinates are particularly suited for spheres and hemispheres:

$$
\begin{cases}
x = \sin \phi \cos \theta \\
y = \sin \phi \sin \theta \\
z = \cos \phi
\end{cases}
$$

where:

- ϕ is the polar (colatitude) angle, measured from the positive z-axis, ranging from 0 to $\pi/2$ for the upper hemisphere.
- θ is the azimuthal angle in the xy-plane, ranging from 0 to 2π.

Since we're dealing with the upper hemisphere:

$$
0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi
$$

Setting Up the Surface Integral $\iint_{S} z \, ds$

Understanding the Integrand and the Surface Element ds

The integrand is z, which in spherical coordinates becomes:

$$
z = \cos \phi
$$

The surface element ds on a sphere can be derived from the parameterization:

$$
ds = |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\phi \, d\theta
$$

where $\mathbf{r}(\phi, \theta)$ is the position vector:

$$
\mathbf{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)
$$

Calculating the derivatives:

$$
\mathbf{r}_\phi = (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi)
$$
$$
\mathbf{r}_\theta = (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0)
$$

The cross product:

$$
\mathbf{r}_\phi \times \mathbf{r}_\theta = (\sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \sin \phi \cos \phi)
$$

The magnitude:

$$
|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{ (\sin^2 \phi \cos \theta)^2 + (\sin^2 \phi \sin \theta)^2 + (\sin \phi \cos \phi)^2 }
$$

Simplify:

$$

= \sqrt{ \sin^4 \phi \cos^2 \theta + \sin^4 \phi \sin^2 \theta + \sin^2 \phi \cos^2 \phi }
\]

\[
= \sqrt{ \sin^4 \phi (\cos^2 \theta + \sin^2 \theta) + \sin^2 \phi \cos^2 \phi }
\]

Since \(\cos^2 \theta + \sin^2 \theta = 1 \):

\[
= \sqrt{ \sin^4 \phi + \sin^2 \phi \cos^2 \phi }
\]

Factor \(\sin^2 \phi \):

\[
= \sqrt{ \sin^2 \phi (\sin^2 \phi + \cos^2 \phi) } = \sqrt{ \sin^2 \phi \times 1 } = \sin \phi
\]

Thus, the surface element:

\[
ds = |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\phi \, d\theta = \sin \phi \, d\phi \, d\theta
\]

Formulating the Integral

Putting all elements together:

\[
\boxed{
\iint_{S} z \, ds = \int_{0}^{2\pi} \int_{0}^{\pi/2} z(\phi) \times |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\phi \, d\theta
}
\]

Substituting \(z = \cos \phi \) and \(ds = \sin \phi \, d\phi \, d\theta \):

\[
\iint_{S} z \, ds = \int_{0}^{2\pi} \int_{0}^{\pi/2} \cos \phi \times \sin \phi \, d\phi \, d\theta
\]

Evaluating the Integral

Step 1: Integrate with respect to ϕ

The inner integral:

$$
I_\phi = \int_{0}^{\pi/2} \cos \phi \sin \phi \, d\phi
$$

This integral can be handled via substitution:

Let:

$$
u = \sin \phi \Rightarrow du = \cos \phi \, d\phi
$$

Change limits:

- When $\phi = 0$, $u = 0$
- When $\phi = \pi/2$, $u = 1$

Thus:

$$
I_\phi = \int_{u=0}^{1} u \, du = \left[\frac{u^2}{2} \right]_0^1 = \frac{1}{2}
$$

Step 2: Integrate with respect to θ

Since the integrand is independent of θ, the outer integral simplifies to:

$$
I_\theta = \int_{0}^{2\pi} d\theta = 2\pi
$$

Final Computation and Result

Multiplying the results of the two integrals:

$$
\iint_{S} z \, ds = I_\theta \times I_\phi = 2\pi \times \frac{1}{2} = \pi
$$

Therefore:

$$
\boxed{
\iint_{S} z \, ds = \pi
}
$$

Insights and Physical Interpretation

This evaluation yields an elegant and concise result, emphasizing the symmetry and inherent geometric properties of the hemisphere. Specifically, integrating \(z \) over the upper hemisphere with respect to surface area essentially computes a weighted average height, scaled by the surface area.

Physical interpretation: If \(z \) represents height or potential in a physical system, this integral quantifies the total "height-weighted" surface measure—an insight valuable in fields like physics, engineering, and environmental sciences.

Summary of Key Steps

- Recognized the surface as a hemisphere and chose spherical coordinates for parameterization.
- Derived the surface element \(ds = \sin \phi \, d\phi \, d\theta \).
- Expressed the integrand \(z \) in spherical coordinates as \(\cos \phi \).
- Set up the double integral over \(\phi \) and \(\theta \).
- Evaluated the inner integral using substitution and the outer integral straightforwardly.
- Arrived at the final answer: \(\pi \).

Conclusion

Evaluating the surface integral \(\i

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