Evaluate The Triple Integral Exdv Where E Is The Solid Bounded By The Paraboloid X=10y2 10z2 And X=10.

Evaluate The Triple Integral Exdv Where E Is The Solid Bounded By The Paraboloid X=10y² + 10z² And X=10.

Introduction

In multivariable calculus, the evaluation of triple integrals plays a crucial role in understanding the volume and other physical properties of three-dimensional regions. The problem at hand involves computing the triple integral $\iiint_E x \, dv$, where E is a solid bounded by the paraboloid $x=10y^2 + 10z^2$ and the plane $x=10$. This is a classic example of integrating over a solid region bounded by quadratic surfaces, which often requires converting the integral into a more convenient coordinate system, such as cylindrical or spherical coordinates, to simplify the process.

Understanding how to evaluate such integrals is fundamental in applications across physics, engineering, and mathematics. These include calculating mass, center of mass, moments of inertia, and other physical quantities where the region of integration is defined by quadratic boundaries.

In this article, we will explore the steps involved in setting up and evaluating the triple integral over the specified region. We will analyze the geometric shape of the solid, choose an appropriate coordinate system, and perform the integration systematically to arrive at the final result. This example not only demonstrates the calculation process but also highlights important techniques in multivariable calculus.

Geometric Description of the Region E

The Boundaries

The region E is bounded by:

- The paraboloid $x=10y^2 + 10z^2$
- The plane $x=10$

These surfaces intersect to form the solid region of integration.

Visualizing the Region

To better understand the shape, consider the following:

- The paraboloid opens along the positive x-axis since $x=10(y^2 + z^2)$.
- For $x=0$, the paraboloid passes through the origin.
- As x increases, the cross-section of the paraboloid in the yz-plane is a circle of radius $\sqrt{x/10}$.

The plane $x=10$ intersects the paraboloid where:

$$
10 = 10(y^2 + z^2) \Rightarrow y^2 + z^2 = 1
$$

This indicates that the intersection is a circle of radius 1 in the yz-plane at $x=10$.

Summary of the Solid E

- The region starts from the vertex at the origin, extending along the x-axis up to $x=10$.

- For each fixed x in $[0, 10]$, the cross-section in the yz-plane is a circle:

$$
y^2 + z^2 \leq \frac{x}{10}
$$

This description suggests the region is a paraboloid "filling up" from the origin to the plane $x=10$.

Choosing the Coordinate System for Integration

Given the symmetry and the nature of the boundaries, cylindrical coordinates are well-suited for integrating over the region E.

Cylindrical Coordinates

Recall the transformations:

$$
\begin{cases}
y = r \cos \theta \\
z = r \sin \theta \\
dv = dx \, dy \, dz = dx \, r \, dr \, d\theta
\end{cases}
$$

The variables range as follows:

- $r \geq 0$, $\theta \in [0, 2\pi)$
- x varies from 0 to 10

- For each fixed x, the cross-section in the yz-plane is $r \leq \sqrt{\frac{x}{10}}$

Setting Up the Integral

The integrand is x, and the differential volume element in cylindrical coordinates is $dv = dx \, r \, dr \, d\theta$.

The limits are:

- $x: 0 \to 10$
- $\theta: 0 \to 2\pi$
- $r: 0 \to \sqrt{\frac{x}{10}}$

Thus, the triple integral becomes:

$$
\iiint_E x \, dv = \int_{x=0}^{10} \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{\frac{x}{10}}} x \, r \, dr \, d\theta \, dx
$$

Evaluating the Integral

Step 1: Integrate with respect to r

$$
\int_0^{\sqrt{\frac{x}{10}}} x \, r \, dr = x \int_0^{\sqrt{\frac{x}{10}}} r \, dr = x \left[\frac{r^2}{2} \right]_0^{\sqrt{\frac{x}{10}}} = x \times \frac{1}{2} \times \left(\frac{x}{10} \right) = \frac{x^2}{20}
$$

\]

Step 2: Integrate with respect to \(\theta\)

\[
\int_0^{2\pi} d\theta = 2\pi
\]

Step 3: Integrate with respect to \(x\)

Now, the integral reduces to:

\[
\iiint_E x \, dv = \int_0^{10} \left(\frac{x^2}{20} \times 2\pi \right) dx = \frac{2\pi}{20} \int_0^{10} x^2 \, dx = \frac{\pi}{10} \left[\frac{x^3}{3} \right]_0^{10} = \frac{\pi}{10} \times \frac{1000}{3} = \frac{\pi}{10} \times \frac{1000}{3}
\]

Simplify:

\[
\frac{\pi}{10} \times \frac{1000}{3} = \frac{1000 \pi}{30} = \frac{100 \pi}{3}
\]

Final Result

\[
\boxed{
\iiint_E x \, dv = \frac{100 \pi}{3}

}

\]

This is the value of the triple integral of x over the solid E bounded by the paraboloid $x=10y^2 + 10z^2$ and the plane $x=10$.

Additional Insights and Applications

Physical Interpretations

Evaluating integrals like $\iiint_E x \, dv$ can be interpreted as calculating the first moment of the region E about the yz-plane, which relates to the center of mass when considering a uniform density distribution.

Variations and Generalizations

- Changing the function x to other functions allows for computing different physical quantities.
- Modifying the bounds or the surfaces can model different physical or geometric scenarios.
- Extending the method to more complex surfaces involves similar steps, often requiring coordinate transformations to simplify the boundaries.

Importance in Engineering and Physics

Such integrals are fundamental in:

- Calculating mass, centroid, and moments of inertia of 3D objects.
- Analyzing heat distribution within a solid.
- Determining charge or mass distribution in physics.

Conclusion

The evaluation of the triple integral $\iiint_E x \, dv$ over the solid bounded by the paraboloid $x=10y^2 + 10z^2$ and the plane $x=10$ exemplifies the power of coordinate transformations and methodical setup in multivariable calculus. By recognizing the symmetry and the geometric shape, selecting the cylindrical coordinate system simplifies the integration process significantly. The final result, $\frac{100 \pi}{3}$, provides a precise measure related to the volume and distribution within this paraboloidal solid.

Understanding and mastering such techniques are essential for tackling complex volume integrals in various scientific and engineering fields, enabling precise analysis and problem-solving in three-dimensional spaces.

Keywords for SEO Optimization

- Triple integral evaluation
- Solid bounded by paraboloid and plane
- Computing volume in multivariable calculus
- Cylindrical coordinates in integration
- Paraboloid region integration
- Physical applications of triple integrals
- Volume integral techniques
- Mathematical methods for 3D regions
- Paraboloid and plane intersection
- Calculus problems on bounded regions

Frequently Asked Questions

What is the solid region E bounded by in the given triple integral problem?

The solid region E is bounded by the paraboloid $X = 10y^2 + 10z^2$ and the plane $X = 10$.

How can we set up the limits of integration for the triple integral over region E?

Since the paraboloid is symmetric in y and z, we can use cylindrical coordinates with X as the radial variable, setting X from 0 to 10 and y, z satisfying $y^2 + z^2$ □ X/10.

What coordinate system is most suitable for evaluating the triple integral over the solid E?

Cylindrical coordinates are most suitable because of the circular symmetry of the paraboloid in the y-z plane.

How do you express the differential volume element dV in cylindrical coordinates for this problem?

In cylindrical coordinates (r, □, x), dV = r dr d□ dx, where r = □($y^2 + z^2$).

What is the step-by-step approach to evaluate the integral □□□_E x dV?

First, express x, y, z in cylindrical coordinates; then set up the limits: x from 0 to 10, r from 0 to x/10, □ from 0 to 2□; substitute into the integral, and integrate iteratively over r, □, and x.

What is the significance of symmetry in simplifying the calculation of the triple integral over E?

Symmetry about the y and z axes allows the integral over the angular coordinate □ to be simplified, often reducing the integral to a multiple of 2□, making the calculation more straightforward.

Additional Resources

Evaluate The Triple Integral Exdv Where E Is The Solid Bounded By The Paraboloid $X=10y^2 + 10z^2$ And X=10

In the realm of multivariable calculus, triple integrals serve as powerful tools to compute volumes, masses, and other quantities over three-dimensional regions. Among the many intriguing problems encountered, evaluating integrals over regions bounded by paraboloids stands out for both its geometric elegance and analytical challenge. Today, we delve into a classic example: Evaluate the triple integral \(\iiint_E x \, dv \), where \(E \) is the solid bounded by the paraboloid \(x = 10 y^2 + 10 z^2 \) and the plane \(x = 10 \). This problem not only offers a rich application of coordinate transformations but also provides insight into how symmetry and bounds influence the evaluation process.

Understanding the Geometric Region \(E \)

Before jumping into the calculations, it’s essential to comprehend the geometric shape defined by the inequalities.

The Paraboloid:

\[

$$
x = 10 y^2 + 10 z^2
$$

This is a standard paraboloid opening along the positive x-axis, with its vertex at the origin. For each fixed x, the cross-section in the y-z plane is a circle:

$$
y^2 + z^2 = \frac{x}{10}
$$

The Bounding Plane:

$$
x = 10
$$

This plane intersects the paraboloid at the point where:

$$
x = 10 y^2 + 10 z^2 = 10
$$

$$
\Rightarrow y^2 + z^2 = 1
$$

So, at $x=10$, the cross-section in the y-z plane is the circle $y^2 + z^2 = 1$.

Region E:

The solid E is thus bounded between the paraboloid $x=10 y^2 + 10 z^2$ (which opens outward from the origin) and the plane $x=10$. For each x in the interval:

$$
x \in [0, 10]
$$

the cross-section in the y-z plane is a disk:

$$
y^2 + z^2 \leq \frac{x}{10}
$$

This description suggests that the entire solid is a paraboloid cap extending from the vertex at the origin up to the plane $x=10$, with a varying radius in the y-z plane.

Setting Up the Integral — Coordinate System Choice

Given the symmetry of the paraboloid $x = 10 y^2 + 10 z^2$, switching to a cylindrical coordinate system simplifies the evaluation significantly.

Cylindrical Coordinates:

$$
\begin{cases}
y = r \cos \theta \\
z = r \sin \theta \\
\text{where } r \geq 0, \quad \theta \in [0, 2\pi]
\end{cases}
$$

Expressing x:

$$
x = 10 r^2
$$

Volume Element:

$$
dv = dx \, dy \, dz = dx \times r \, dr \, d\theta
$$

since in cylindrical coordinates, $dy\,dz = r\, dr\, d\theta$.

Determining the Limits of Integration

For x:

- At the base: x varies from the vertex $x=0$ up to the plane $x=10$.

- For a fixed x, the cross-section in the y-z plane is a disk with radius:

$$
r = \sqrt{\frac{x}{10}}
$$

- The maximum value of r at each x:

$$
r_{max} = \sqrt{\frac{x}{10}}
$$

For r:

- For each x, r varies from 0 to $\sqrt{\frac{x}{10}}$.

For θ:

- The angle θ spans the full circle:

$$
\theta \in [0, 2\pi]
$$

Expressing the Integral

The original integral:

$$
I = \iiint_E x \, dv
$$

becomes:

$$
I = \int_{x=0}^{10} \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{\frac{x}{10}}} x \times r \, dr \, d\theta \, dx
$$

Note that x is a function of the fixed coordinate, and the integrand includes x.

Evaluating the Integral Step-by-Step

Step 1: Integrate with respect to r:

$$
\int_{0}^{\sqrt{\frac{x}{10}}} x \times r \, dr = x \int_{0}^{\sqrt{\frac{x}{10}}} r \, dr
= x \left[\frac{r^2}{2} \right]_0^{\sqrt{\frac{x}{10}}} = x \times \frac{1}{2} \times \frac{x}{10} =
\frac{x^2}{20}
$$

since:

$$
r^2 = \left(\sqrt{\frac{x}{10}} \right)^2 = \frac{x}{10}
$$

Step 2: Integrate with respect to θ:

$$
\int_0^{2\pi} d\theta = 2\pi
$$

which is straightforward.

Step 3: Integrate with respect to x:

Putting it all together, the integral reduces to:

$$
I = \int_0^{10} \left(\frac{x^2}{20} \times 2\pi \right) dx = \frac{2\pi}{20} \int_0^{10} x^2 \, dx =
\frac{\pi}{10} \int_0^{10} x^2 \, dx
$$

\]

Calculating:

$$
\int_0^{10} x^2 \, dx = \left[\frac{x^3}{3} \right]_0^{10} = \frac{10^3}{3} = \frac{1000}{3}
$$

Thus, the value of the integral:

$$
I = \frac{\pi}{10} \times \frac{1000}{3} = \frac{\pi \times 1000}{10 \times 3} = \frac{1000 \pi}{30} = \frac{100 \pi}{3}
$$

Final Result and Interpretation

The value of the triple integral is:

$$
\boxed{
\iiint_E x \, dv = \frac{100 \pi}{3}
}
$$

This result reflects the weighted volume of the region E, considering the x-coordinate as a weight function. Notably, the symmetry and the choice of cylindrical coordinates streamlined the process, transforming a seemingly complex three-dimensional integral into manageable steps.

Broader Implications and Applications

This example demonstrates the power of coordinate transformations in multivariable calculus. When regions exhibit symmetry, especially circular symmetry in the y-z plane, cylindrical coordinates often provide an elegant pathway for evaluation. The problem also underscores the importance of understanding the geometric bounds—here, the paraboloid and plane—to set accurate limits of integration.

In practical applications, such calculations could relate to moments of inertia, mass distributions, or flux calculations within bounded regions in physics and engineering. The approach outlined here is adaptable to more complex shapes, emphasizing the importance of visualization and choosing appropriate coordinate systems.

Summary

- Region E: Bounded by the paraboloid $x=10 y^2 + 10 z^2$ and the plane $x=10$.

- Coordinate system: Cylindrical coordinates simplify the integral owing to symmetry.

- Limits:

- x from 0 to 10.

- r from 0 to $\sqrt{\frac{x}{10}}$.

- θ from 0 to 2π.

- Integral evaluation:

- Inner integral yields $\frac{x^2}{20}$.

- Outer integral evaluates to $\frac{100 \pi}{3}$.

This comprehensive analysis not only provides the numerical result but also illustrates the methodology and reasoning essential for tackling similar three-dimensional integral problems in advanced calculus.

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