Express The Function F(xy) = X 2 + 3y 2 9x 9y + 26 As Taylors Series Expansion About The Point (1, 2).

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Introduction to Taylor Series Expansions

Understanding the Taylor series expansion of a function provides a powerful method for approximating complex functions using polynomials. This technique is especially useful in calculus and mathematical analysis, as it allows for the approximation of functions near a specific point with increasing accuracy by including higher-order derivatives.

In this article, we focus on expressing the function F(x, y) = $x^2 + 3y^2 - 9x - 9y + 26$ as a Taylor series expansion about the point (1, 2). This process involves calculating the function's derivatives at the point of expansion and constructing the polynomial that approximates the function in the vicinity of (1, 2).

Understanding the Function F(x, y)

Before delving into the expansion process, let's analyze the given function:

$$F(x, y) = x^2 + 3y^2 - 9x - 9y + 26$$

This function is a quadratic form involving two variables, x and y. It resembles the general form of a conic section, such as a parabola or ellipse, depending on the coefficients.

Key features of the function include:

- Quadratic terms in x and y
- Linear terms in x and y
- A constant term

Our goal is to approximate F(x, y) near the point (1, 2) using the Taylor series expansion.

Step-by-Step Process of Taylor Series Expansion

The Taylor series expansion of a multivariable function F(x, y) about a point (a, b) is given by:

$$
F(x, y) \approx F(a, b) + \nabla F(a, b) \cdot \begin{bmatrix} x - a \\ y - b \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x - a & y - b \end{bmatrix} H_F(a, b) \begin{bmatrix} x - a \\ y - b \end{bmatrix} + \dots
$$

where:

- $F(a, b)$ is the function value at the expansion point.
- $\nabla F(a, b)$ is the gradient vector at (a, b).
- $H_F(a, b)$ is the Hessian matrix at (a, b).
- Higher-order terms involve derivatives of order three and above (which we'll omit for a second-order approximation).

Let's proceed step-by-step.

---

1. Calculate $F(1, 2)$

Evaluate the function at the point (1, 2):

$$
F(1, 2) = (1)^2 + 3(2)^2 - 9(1) - 9(2) + 26
$$
$$
= 1 + 3 \times 4 - 9 - 18 + 26
$$
$$
= 1 + 12 - 9 - 18 + 26
$$
$$
= (1 + 12) - 9 - 18 + 26 = 13 - 9 - 18 + 26
$$
$$
= (13 - 9) - 18 + 26 = 4 - 18 + 26 = (4 - 18) + 26 = -14 + 26 = 12
$$

Result: $F(1, 2) = 12$

---

2. Compute the Gradient $\nabla F(x, y)$ at (1, 2)

The gradient vector consists of the partial derivatives:

$$
\nabla F = \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right]
$$

Calculate each partial derivative:

- $\frac{\partial F}{\partial x} = 2x - 9$
- $\frac{\partial F}{\partial y} = 6y - 9$

Evaluate at (1, 2):

$$
\frac{\partial F}{\partial x} (1, 2) = 2(1) - 9 = 2 - 9 = -7
$$
$$
\frac{\partial F}{\partial y} (1, 2) = 6(2) - 9 = 12 - 9 = 3
$$

Gradient at (1, 2):

$$
\nabla F(1, 2) = \begin{bmatrix} -7 \\ 3 \end{bmatrix}
$$

---

3. Compute the Hessian Matrix $H_F(x, y)$ at (1, 2)

The Hessian matrix contains second derivatives:

$$
H_F = \begin{bmatrix}
\frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\
\frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2}
\end{bmatrix}
$$

Calculate each:

- $\frac{\partial^2 F}{\partial x^2} = 2$
- $\frac{\partial^2 F}{\partial y^2} = 6$
- $\frac{\partial^2 F}{\partial x \partial y} = 0$ (since the cross derivatives are zero for this quadratic form)

At any point, the Hessian remains constant:

$$
H_F = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}
$$

Constructing the Taylor Series Expansion

Using the calculated components, the second-order Taylor series expansion of F(x, y) about (1, 2) is:

$$
F(x, y) \approx F(1, 2) + \nabla F(1, 2) \cdot \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x - 1 & y - 2 \end{bmatrix} H_F \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix}
$$

Substitute the known values:

$$
F(x, y) \approx 12 + \begin{bmatrix} -7 & 3 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x - 1 & y - 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix}
$$

Final Taylor Series Expression

Expanding the terms:

- Linear term:

$$
-7(x - 1) + 3(y - 2)
$$

- Quadratic term:

$$
\frac{1}{2} \left[2(x - 1)^2 + 6(y - 2)^2 \right] = (x - 1)^2 + 3(y - 2)^2
$$

Putting everything together:

$$
\boxed{
F(x, y) \approx 12 - 7(x - 1) + 3(y - 2) + (x - 1)^2 + 3(y - 2)^2
}
$$

This quadratic polynomial serves as an accurate approximation of the original function F(x, y) near the point (1, 2).

Applications and Significance of the Taylor Series Expansion

The Taylor series expansion has numerous practical applications across various fields:

- Numerical Analysis: Facilitates the approximation of functions where exact calculations are complex or impossible.
- Physics and Engineering: Used in modeling systems, such as in control theory and mechanics, where small deviations are analyzed.
- Optimization: Assists in finding local minima or maxima by examining the behavior of functions near critical points.
- Computer Science: Enables efficient algorithms for functions evaluation, especially in graphics and simulations.

Understanding how to derive and utilize Taylor series expansions, especially in multiple variables, is essential for mathematicians, engineers, and scientists working with complex models.

Conclusion

Expressing the function $F(x, y) = x^2 + 3y^2 - 9x - 9y + 26$ as a Taylor series expansion about the point (1, 2) provides a powerful approximation tool. The process involves calculating the function's value, gradient, and Hessian at the point, then constructing a quadratic polynomial that approximates the original function in the neighborhood.

The resulting expansion:

\[
F(x, y) \approx 12 - 7(x - 1) + 3(y - 2) + (x -

Frequently Asked Questions

What is the goal of expanding the function F(x, y) = $x^2 + 3y^2 - 9x - 9y + 26$ into a Taylor series about the point (1, 2)?

The goal is to approximate the function near the point (1, 2) using its derivatives at that point, providing a polynomial that simplifies analysis and computation of the function's behavior locally.

How do we find the Taylor series expansion of F(x, y) around (1, 2)?

We compute the function's value, first derivatives, and second derivatives at (1, 2), then substitute these into the Taylor series formula for functions of two variables up to the desired degree.

What are the first step calculations needed for the Taylor series of F(x, y) about (1, 2)?

Calculate F(1, 2), the partial derivatives Fx and Fy at (1, 2), and the second derivatives Fxx, Fyy, and Fxy at that point.

What is the value of the function F(x, y) at the point (1, 2)?

$F(1, 2) = (1)^2 + 3(2)^2 - 9(1) - 9(2) + 26 = 1 + 34 - 9 - 18 + 26 = 1 + 12 - 9 - 18 + 26 = 12$.

How are the partial derivatives Fx and Fy at (1, 2) calculated?

Fx = derivative of F with respect to x: 2x - 9; at (1, 2): Fx = 21 - 9 = -7. Fy = derivative of F with respect to y: 6y - 9; at (1, 2): Fy = 62 - 9 = 12 - 9 = 3.

What are the second derivatives of F at (1, 2) and their values?

Fxx = 2, Fyy = 6, Fxy = 0; these are constants since the second derivatives are independent of x and y.

What is the final Taylor series expansion of F(x, y) about (1, 2) up to second order?

$F(x, y) \approx 12 + (-7)(x - 1) + 3(y - 2) + (1/2)[2(x - 1)^2 + 0 + 6(y - 2)^2] = 12 - 7(x - 1) + 3(y - 2) + (x - 1)^2 + 3(y - 2)^2$.

Why is Taylor series expansion useful for the function F(x, y) near the point (1, 2)?

It provides a simplified polynomial approximation that makes analyzing the function's local behavior, derivatives, and changes easier, especially for small deviations from the point (1, 2).

Additional Resources

Taylor Series Expansion

Introduction

In the realm of mathematical analysis and multivariable calculus, the Taylor series expansion stands as a powerful tool to approximate functions near a specific point. Its application spans across various fields such as engineering, physics, computer science, and applied mathematics, providing insights into the local behavior of complex functions. Today, we delve into a thorough exploration of the function:

$$
F(x, y) = x^2 + 3 y^2 - 9 x y + 26
$$

and examine its Taylor series expansion about the point $(1, 2)$. This exploration not only uncovers the step-by-step methodology behind the expansion but also highlights its significance as an analytical instrument to approximate and understand functions in the vicinity of a point of interest.

Understanding the Function $F(x, y)$

Before embarking on the expansion, it's essential to understand the structure and properties of the function:

$$
F(x, y) = x^2 + 3 y^2 - 9 x y + 26
$$

Nature of the Function

- Type: Quadratic function in two variables.
- Components:
- x^2: a quadratic term in x.
- $3 y^2$: quadratic term in y, scaled by 3.
- $-9 x y$: a mixed term representing the interaction between x and y.
- $+26$: a constant offset.

The function is a quadratic form, which can be represented in matrix notation:

$$
F(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x} + c
$$
where

$$
\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}
$$

and

$$
Q = \begin{bmatrix} 1 & -\frac{9}{2} \\ -\frac{9}{2} & 3 \end{bmatrix}
$$

This matrix form indicates that F is a quadratic form with symmetric matrix Q. The symmetry ensures the function is well-behaved and convex in many regions.

Significance of Expansion at $(1, 2)$

Choosing the point $(1, 2)$ is strategic for local approximation, especially in applications like optimization, where understanding the behavior near a specific point can guide decision-making or further analysis.

Step 1: Calculating the Necessary Derivatives

The core of the Taylor series expansion involves derivatives of the function evaluated at the point of expansion, $(x_0, y_0) = (1, 2)$.

1. First-Order Partial Derivatives

$$
F_x = \frac{\partial F}{\partial x} = 2x - 9 y
$$
$$
F_y = \frac{\partial F}{\partial y} = 6 y - 9 x
$$

2. Second-Order Partial Derivatives

$$
F_{xx} = \frac{\partial^2 F}{\partial x^2} = 2
$$
$$
F_{yy} = \frac{\partial^2 F}{\partial y^2} = 6
$$
$$
F_{xy} = F_{yx} = \frac{\partial^2 F}{\partial x \partial y} = -9
$$

Note that the mixed partial derivatives are equal, satisfying Clairaut's theorem.

3. Evaluating Derivatives at $(1, 2)$

$$
F_x(1, 2) = 2(1) - 9(2) = 2 - 18 = -16
$$
$$
F_y(1, 2) = 6(2) - 9(1) = 12 - 9 = 3
$$

$$
F_{xx}(1, 2) = 2
$$
$$
F_{yy}(1, 2) = 6
$$
$$
F_{xy}(1, 2) = -9
$$

Step 2: Constructing the Taylor Series Expansion

The multivariable Taylor series expansion of $F(x, y)$ about the point (x_0, y_0) up to the second order is given by:

\[

F(x, y) \approx F(x_0, y_0) + F_x(x_0, y_0) (x - x_0) + F_y(x_0, y_0) (y - y_0) + \frac{1}{2} \left[F_{xx}(x_0, y_0) (x - x_0)^2 + 2 F_{xy}(x_0, y_0) (x - x_0)(y - y_0) + F_{yy}(x_0, y_0) (y - y_0)^2 \right]
\]

1. Calculating \(F(x_0, y_0)\)

\[
F(1, 2) = (1)^2 + 3 (2)^2 - 9(1)(2) + 26 = 1 + 3 \times 4 - 18 + 26 = 1 + 12 - 18 + 26 = 21
\]

2. Substituting known derivatives

\[
\boxed{
\begin{aligned}
F(x, y) \approx\ & 21 + (-16)(x - 1) + 3 (y - 2) \\
& + \frac{1}{2} \left[2 (x - 1)^2 + 2 \times (-9) (x - 1)(y - 2) + 6 (y - 2)^2 \right]
\end{aligned}
}
\]

Simplify the second-order terms:

\[
\frac{1}{2} \left[2 (x - 1)^2 - 18 (x - 1)(y - 2) + 6 (y - 2)^2 \right]
\]

which simplifies to:

\[
(x - 1)^2 - 9 (x - 1)(y - 2) + 3 (y - 2)^2
\]

3. Final Taylor Series Approximation

\[
\boxed{
F(x, y) \approx 21 - 16 (x - 1) + 3 (y - 2) + (x - 1)^2 - 9 (x - 1)(y - 2) + 3 (y - 2)^2
}
\]

This quadratic approximation offers an insightful local snapshot of the function's behavior near \((1, 2)\). It can be used to estimate the function's value for points close to this expansion point, as well as to analyze the local curvature and minima.

Deeper Analysis: Significance and Applications

Why Use Taylor Series?

- Local Approximation: Simplifies complex functions into polynomials, making calculations more manageable.
- Optimization: Helps locate local minima or maxima through the quadratic form of the expansion.
- Sensitivity Analysis: Understand how small changes in inputs affect the output.
- Numerical Methods: Serve as the foundation for algorithms in numerical differentiation and integration.

Advantages of the Second-Order Expansion

- Captures the curvature of the function, providing a more accurate approximation than linear terms alone.
- Enables the identification of the nature of critical points (minima, maxima, saddle points) via the Hessian matrix.

Limitations

- Valid only near $(1, 2)$; as you move further away, the approximation's accuracy diminishes.
- Higher-order derivatives are needed for more precise approximations over larger regions.

Practical Implications in Real-World Contexts

Imagine an engineer assessing the stress distribution in a material modeled by such a quadratic function. The Taylor expansion near a point allows for quick estimations of stress responses to small perturbations in design parameters. Similarly, in machine learning, such local approximations underpin gradient-based optimization algorithms like Newton's method, which rely on second derivatives to find minima efficiently.

Summary and Final Thoughts

The Taylor series expansion of the function $F(x, y) = x^2 + 3 y^2 - 9 x y + 26$ around the point $(1, 2)$ exemplifies the power and elegance of multivariable calculus. Through meticulous derivative calculations and systematic substitution, we've arrived at a quadratic approximation that encapsulates the local behavior of F. This expansion not only provides immediate practical utility but also deepens our understanding of the function's geometry and potential applications.

By mastering such expansions, mathematicians, engineers, and scientists gain a versatile tool for simplifying complex functions, optimizing systems, and analyzing responses in a multitude of scientific and engineering domains. The process underscores the importance of derivatives and the local linear/quadratic models in approximating and understanding the world through mathematical lenses.

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