Find D Dt [r(t) U(t)] And D Dt [r(t) U(t)] In Two Different Ways. R(t) = Cos Ti Sin Tj Tk, U(t) = J Tk

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Introduction

In the realm of vector calculus, differentiation of vector functions plays a pivotal role in understanding the behavior of physical systems, especially in fields such as physics, engineering, and robotics. One common problem involves finding the derivative of the cross product of two vector functions with respect to a parameter, typically time t.

This article aims to guide you through the process of finding the derivative of $\mathbf{r}(t) \times \mathbf{U}(t)$ with respect to t, using two different methods. Specifically, we will analyze the vector functions:

- $\mathbf{r}(t) = \cos t \, \mathbf{i} + \sin t \, \mathbf{j} + t \, \mathbf{k}$
- $\mathbf{U}(t) = t \, \mathbf{k}$

Our goal is to compute:

$$\frac{d}{dt} \left[\mathbf{r}(t) \times \mathbf{U}(t)\right]$$

using both the product rule for derivatives of cross products and an alternative approach, to illustrate different techniques for solving such problems.

Understanding the Vector Functions

Before diving into differentiation, let's understand the given vector functions:

The Vector Function $\mathbf{r}(t)$

$$\mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$$

This describes a helix-like path in 3D space, with components that oscillate in the xy-plane and linearly increase along the z-axis as t

increases.

The Vector Function $\mathbf{U}(t)$

$$
\mathbf{U}(t) = t\, \mathbf{k}
$$

A simple linear vector along the z-axis, with magnitude increasing uniformly with t.

Method 1: Applying the Product Rule for Cross Products

The Vector Calculus Product Rule

The derivative of the cross product of two vector functions $\mathbf{A}(t)$ and $\mathbf{B}(t)$ is given by:

$$
\frac{d}{dt} \left[\mathbf{A}(t) \times \mathbf{B}(t) \right] = \frac{d\mathbf{A}}{dt} \times \mathbf{B}(t) + \mathbf{A}(t) \times \frac{d\mathbf{B}}{dt}
$$

This rule is analogous to the product rule for scalar functions, extended to vectors and their cross products.

Step 1: Compute $\frac{d\mathbf{r}(t)}{dt}$

Differentiate each component:

$$
\frac{d\mathbf{r}(t)}{dt} = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + 1\, \mathbf{k}
$$

Step 2: Compute $\frac{d\mathbf{U}(t)}{dt}$

Since $\mathbf{U}(t) = t\, \mathbf{k}$:

$$
\frac{d\mathbf{U}(t)}{dt} = 0\, \mathbf{i} + 0\, \mathbf{j} + 1\, \mathbf{k}
$$

Step 3: Compute $\mathbf{r}(t) \times \mathbf{U}(t)$

Calculate the cross product $\mathbf{r}(t) \times \mathbf{U}(t)$:

$$

\mathbf{r}(t) \times \mathbf{U}(t) =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos t & \sin t & t \\
0 & 0 & t \\
\end{vmatrix}
\]

Expanding the determinant:

\[
\mathbf{i} \left(\sin t \times t - t \times 0 \right) - \mathbf{j} \left(\cos t \times t - t \times 0 \right) + \mathbf{k} \left(\cos t \times 0 - \sin t \times 0 \right)
\]

Simplifies to:

\[
\mathbf{i} (t \sin t) - \mathbf{j} (t \cos t) + \mathbf{k} (0)
\]

Thus:

\[
\boxed{
\mathbf{r}(t) \times \mathbf{U}(t) = t \sin t\, \mathbf{i} - t \cos t\, \mathbf{j}
}
\]

Step 4: Derive \(\frac{d}{dt} [\mathbf{r}(t) \times \mathbf{U}(t)] \)

Differentiate each component:

\[
\frac{d}{dt} \left(t \sin t \right) = \sin t + t \cos t
\]
\[
\frac{d}{dt} \left(- t \cos t \right) = - \left(\cos t - t \sin t \right) = - \cos t + t \sin t
\]

Hence:

\[
\frac{d}{dt} \left[\mathbf{r}(t) \times \mathbf{U}(t) \right] = (\sin t + t \cos t) \, \mathbf{i} + (- \cos t + t \sin t) \, \mathbf{j}
\]

Method 2: Using the Derivative Formula for Cross Products

Instead of directly differentiating the cross product, we can re-express the derivative using the product rule for cross products we established earlier:

$$
\frac{d}{dt} \left[\mathbf{r}(t) \times \mathbf{U}(t) \right] = \frac{d\mathbf{r}(t)}{dt} \times \mathbf{U}(t) + \mathbf{r}(t) \times \frac{d \mathbf{U}(t)}{dt}
$$

Step 1: Recall the derivatives computed earlier:

- $\frac{d\mathbf{r}(t)}{dt} = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + \mathbf{k}$
- $\frac{d\mathbf{U}(t)}{dt} = \mathbf{k}$

Step 2: Compute $\frac{d\mathbf{r}(t)}{dt} \times \mathbf{U}(t)$

Calculate:

$$
\left(-\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + \mathbf{k} \right) \times \left(t\, \mathbf{k} \right)
$$

Distribute the cross product:

$$
- \sin t\, \mathbf{i} \times t\, \mathbf{k} + \cos t\, \mathbf{j} \times t\, \mathbf{k} + \mathbf{k} \times t\, \mathbf{k}
$$

Using the standard cross products:

- $\mathbf{i} \times \mathbf{k} = - \mathbf{j}$
- $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
- $\mathbf{k} \times \mathbf{k} = \mathbf{0}$

Calculations:

$$
- \sin t\, t\, (- \mathbf{j}) + \cos t\, t\, \mathbf{i} + 0 = \sin t\, t\, \mathbf{j} + t \cos t\, \mathbf{i}
$$

Rearranged:

$$
t \cos t\, \mathbf{i} + t \sin t\, \mathbf{j}
$$

Step 3: Compute \(\mathbf{r}(t) \times \frac{d \mathbf{U}(t)}{dt} \)

Calculate:

\[
\left(\cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k} \right)
\times \mathbf{k}
\]

Breaking down:

\[
\cos t\, \mathbf{i} \times \mathbf{k} + \sin t\, \mathbf{j} \times \mathbf{k}
+ t\, \mathbf{k} \times \mathbf{k}
\]

Using the cross product identities:

- \(\mathbf{i} \times \mathbf{k} = - \mathbf{j} \)
- \(\mathbf{j} \times \mathbf{k} = \mathbf{i} \)
- \(\mathbf{k} \times \mathbf{k} = 0 \)

Results:

\[
\cos t\, (- \mathbf{j}) + \sin t\, \mathbf{i} + 0 = - \cos t\, \mathbf{j} +
\sin t\, \mathbf{i}
\]

Final Result

Adding the two components obtained:

\[
\left(t \cos t\, \mathbf{i} +

Frequently Asked Questions

What is the derivative of the vector function r(t) = cos t i + sin t j + t k with respect to t?

The derivative D/dt [r(t)] is -sin t i + cos t j + k.

How do you compute the derivative of the cross

product r(t) × U(t) where r(t) = cos t i + sin t j + t k and U(t) = j k?

Using the product rule for cross products: D/dt [r(t) × U(t)] = D r(t)/dt × U(t) + r(t) × D U(t)/dt. Since U(t) = j k (a constant vector), its derivative is zero, so the derivative simplifies to D r(t)/dt × U(t).

What are two different methods to find D/dt [r(t) U(t)] for the given vectors?

Method 1: Use the product rule for derivatives of the scalar or vector functions multiplied by the unit step function U(t). Method 2: Express r(t) U(t) as a piecewise function and differentiate directly, considering the step function's properties.

How does the unit step function U(t) influence the differentiation of r(t) U(t)?

The unit step function U(t) is zero for t < 0 and one for t ≥ 0. When differentiating r(t) U(t), the derivative involves both the derivative of r(t) multiplied by U(t) and a term involving the Dirac delta function at t=0, reflecting the discontinuity.

Can you explain the derivative D/dt [r(t) U(t)] using the product rule?

Yes, D/dt [r(t) U(t)] = D r(t)/dt U(t) + r(t) δ(t), where δ(t) is the Dirac delta function representing the derivative of the step function at t=0.

What is the explicit expression of D/dt [r(t) U(t)] for r(t) = cos t i + sin t j + t k and U(t) = j k?

Since U(t) is constant (j k), the derivative simplifies to D r(t)/dt U(t). Calculating: D r(t)/dt = -sin t i + cos t j + k, so the derivative is (-sin t i + cos t j + k) × (j k).

How do you compute D/dt [r(t) × U(t)] using component-wise differentiation?

Since U(t) = j k is constant, D/dt [r(t) × U(t)] = D r(t)/dt × U(t). First, find D r(t)/dt = -sin t i + cos t j + k, then compute its cross product with j k to get the derivative.

Why is it important to consider the properties of

the step function when differentiating r(t) U(t)?

Because U(t) introduces a discontinuity at t=0, leading to the presence of the Dirac delta function in the derivative. Recognizing this ensures correct calculation of derivatives involving step functions.

Additional Resources

Find D Dt [r(t) U(t)] And D Dt [r(t) U(t)] In Two Different Ways. R(t) = Cos T i + Sin T j + T k, U(t) = T k

The problem of differentiating the cross product of vector functions such as r(t) U(t) is a fundamental topic in vector calculus, with wide applications in physics, engineering, and computer graphics. The specific functions given, r(t) = cos T i + sin T j + T k and U(t) = T k, provide an excellent context to explore different methods of differentiation, especially the product rule for vector functions. This article aims to analyze the differentiation process of D Dt [r(t) U(t)] in detail, presenting two distinct approaches, discussing their features, advantages, and limitations.

Understanding the Problem and the Given Functions

The goal is to compute D Dt [r(t) U(t)] where:

- r(t) = cos T i + sin T j + T k
- U(t) = T k

Here, r(t) is a vector function with a sinusoidal component in the x and y directions and a linear component in the z direction. The function U(t) is a simple vector pointing in the z direction, scaled by T. The problem involves differentiating the cross product of these functions with respect to time t.

Since r(t) and U(t) are vector functions of t, the differentiation involves applying the product rule for vector functions, which can be approached in different ways, as will be elaborated below.

Method 1: Direct Application of the Vector

Product Rule

Principle

The first method involves directly applying the standard product rule for derivatives of the cross product of vector functions:

$$
\frac{d}{dt} [\mathbf{r}(t) \times \mathbf{U}(t)] = \frac{d\mathbf{r}(t)}{dt} \times \mathbf{U}(t) + \mathbf{r}(t) \times \frac{d\mathbf{U}(t)}{dt}
$$

This approach is straightforward, relying on known differentiation rules and properties of the cross product.

Step-by-step Derivation

1. Calculate $\frac{d\mathbf{r}(t)}{dt}$:

Given $\mathbf{r}(t) = \cos T \mathbf{i} + \sin T \mathbf{j} + T \mathbf{k}$, then:

$$
\frac{d\mathbf{r}(t)}{dt} = -\sin T \mathbf{i} + \cos T \mathbf{j} + \mathbf{k}
$$

2. Calculate $\frac{d\mathbf{U}(t)}{dt}$:

Since $\mathbf{U}(t) = T \mathbf{k}$,

$$
\frac{d\mathbf{U}(t)}{dt} = \mathbf{k}
$$

3. Compute the derivatives of the cross product:

$$
\frac{d}{dt} [\mathbf{r}(t) \times \mathbf{U}(t)] = (-\sin T \mathbf{i} + \cos T \mathbf{j} + \mathbf{k}) \times (T \mathbf{k}) + (\cos T \mathbf{i} + \sin T \mathbf{j} + T \mathbf{k}) \times \mathbf{k}
$$

4. Evaluate each cross product:

- First term:

$$
(-\sin T \mathbf{i} + \cos T \mathbf{j} + \mathbf{k}) \times T \mathbf{k} = T [(-\sin T \mathbf{i}) \times \mathbf{k} + (\cos T \mathbf{j}) \times \mathbf{k} + \mathbf{k} \times \mathbf{k}]
$$

Recall:

$$
\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{k} = \mathbf{0}
$$

Thus,

$$
T \left[-\sin T (-\mathbf{j}) + \cos T (\mathbf{i}) + 0 \right] = T [\sin T \mathbf{j} + \cos T \mathbf{i}]
$$

- Second term:

$$
(\cos T \mathbf{i} + \sin T \mathbf{j} + T \mathbf{k}) \times \mathbf{k} = (\cos T \mathbf{i}) \times \mathbf{k} + (\sin T \mathbf{j}) \times \mathbf{k} + T \mathbf{k} \times \mathbf{k}
$$

which evaluates to:

$$
\cos T (-\mathbf{j}) + \sin T (\mathbf{i}) + 0 = -\cos T \mathbf{j} + \sin T \mathbf{i}
$$

5. Combine results:

$$
\frac{d}{dt} [\mathbf{r}(t) \times \mathbf{U}(t)] = T (\sin T \mathbf{j} + \cos T \mathbf{i}) + (-\cos T \mathbf{j} + \sin T \mathbf{i})
$$

6. Final expression:

$$
\boxed{
\frac{d}{dt} [\mathbf{r}(t) \times \mathbf{U}(t)] = (T \cos T + \sin T) \mathbf{i} + (T \sin T - \cos T) \mathbf{j}
}
$$

Features of Method 1:

- Advantages:
- Simple to implement as it directly applies the product rule.
- Clear step-by-step process, suitable for computational approaches.
- Limitations:
- Can be cumbersome for more complex functions or multiple products.
- Requires careful calculation of each derivative and cross product.

Method 2: Using the Properties of the Cross Product and Chain Rule

Principle

The second approach leverages the properties of the cross product and the chain rule, focusing on the idea that the derivative of a cross product can be viewed as a combination of derivatives of the constituent vectors, possibly simplifying calculations.

Instead of directly computing derivatives term-by-term, this method emphasizes identifying derivative patterns, applying vector identities, and exploiting the structure of the functions.

Step-by-step Derivation

1. Express the derivative in a general form:

$$
\frac{d}{dt} [\mathbf{r}(t) \times \mathbf{U}(t)] = \mathbf{\dot{r}}(t) \times \mathbf{U}(t) + \mathbf{r}(t) \times \mathbf{\dot{U}}(t)
$$

which is the same as Method 1's formula, but here we interpret it through the lens of vector identities and properties.

2. Analyze the structure of $\mathbf{r}(t)$ and $\mathbf{U}(t)$:

- $\mathbf{r}(t)$ combines sinusoidal and linear components, which allows for recognizing patterns in derivatives.
- $\mathbf{U}(t)$ is linear in T, simplifying $\mathbf{\dot{U}}(t)$.

3. Identify useful vector identities:

- Cross product derivatives follow the product rule.
- The derivatives of sinusoidal functions involve sine and cosine, which naturally pair with cross products involving unit vectors.

4. Compute derivatives:

- $\mathbf{\dot{r}}(t) = -\sin T \mathbf{i} + \cos T \mathbf{j} + \mathbf{k}$
- $\mathbf{\dot{U}}(t) = \mathbf{k}$

5. Compute the cross products using vector identities:

- Recognize that $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{k} = 0$.
- Use these identities to evaluate each term systematically, possibly employing vector notation for compactness.

6. Simplify the sum:

By grouping terms and using known identities, the derivative can be expressed in a more compact form, emphasizing the geometric interpretation of the cross products and the effect of the derivatives.

Comparison of the Two Methods

Feature	Method 1: Direct Application	Method 2: Using Properties & Identities
Approach	Stepwise, straightforward application of product rule	Emphasizes vector identities and structural understanding
Ease of Use	Simple for basic functions; can become cumbersome for complex functions	More conceptual; helpful for recognizing patterns
Computational Efficiency	Good for straightforward problems	Better for complex functions with repeated structures
Clarity	Clear step-by-step, suitable for beginners	Provides deeper insight into the structure of the problem

Pros of Method 1:

- Easy to follow and implement.
- Suitable for computational tools like MATLAB or Mathematica.
- Good for initial understanding.

Cons of Method 1:

- Can be

Find D Dt R T U T And D Dt R T U T In Two Different Ways R T Cos Ti Sin Tj Tk U T J Tk

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